Math 1

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Matrix

A *matrix* is a set of real or complex numbers (or *elements*) arranged in rows and columns to form a rectangular array.

A matrix having m rows and n columns is called an $m \times n$ (i.e. 'm by n') matrix and is referred to as having *order* $m \times n$.

A matrix is indicated by writing the array within brackets

e.g. $\begin{pmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{pmatrix}$ is a 2×3 matrix, i.e. a '2 by 3' matrix, where

5, 7, 2, 6, 3, 8 are the elements of the matrix.

Note that, in describing the matrix, the number of rows is stated first and the number of columns second.

Double suffix notation: Each element in a matrix has its own particular 'address' or location which can be defined by a system of double suffixes, the first indicating the row, the second the column, thus:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

$\therefore$ a_{23} indicates the element in the second row and third column.

Therefore, in the matrix

$$\begin{pmatrix} 6 & -5 & 1 & -3 \\ 2 & -4 & 8 & 3 \\ 4 & -7 & -6 & 5 \\ -2 & 9 & 7 & -1 \end{pmatrix}$$

The location of -5 is a_{12} ,,, and location of 3 is a_{24}

Matrix notation :

Where there is no ambiguity, a whole matrix can be denoted by a single general element enclosed in brackets, or by a single letter printed in bold type. This is a very neat shorthand and saves much space and writing. For example:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} \text{ can be denoted by } (a_{ij}) \text{ or } (a) \text{ or by } \mathbf{A}.$$

$$\text{Similarly } \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ can be denoted by } (x_i) \text{ or } (x) \text{ or simply by } \mathbf{x}.$$

For an $(m \times n)$ matrix, we use a bold capital letter, e.g. **A**. For a row or column matrix, we use a lower-case bold letter, e.g. **x**. (In handwritten work, we can indicate bold-face type by a wavy line placed under the letter, e.g. $\underset{\sim}{A}$ or $\underset{\sim}{x}$.)

So, if **B** represents a 2×3 matrix, write out the elements b_{ij} in the matrix, using the double suffix notation. This gives . . . .

$$B=\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{pmatrix}$$

Equal Matrices :

By definition, two matrices are said to be equal if corresponding elements throughout are equal. Thus, the two matrices must also be of the same order.

By symbols $A_{m\times n}$ and $B_{m\times n}$ are two matrices then $A_{m\times n}= B_{m\times n}$ if and only if $a_{ij} = b_{ij}$ i=1,2….m , j=1,2…n

EX:

1. A= $\begin{pmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{pmatrix}$ B= $\begin{pmatrix} 1 & -3 & 4 \\ 0 & 5 & -2 \end{pmatrix}$

A=B

1. If A= $\begin{pmatrix} a & b \\ c & d \\ e & f \end{pmatrix}$ = B= $\begin{pmatrix} 1 & -2 \\ 16 & 4 \\ 5 & 11 \end{pmatrix}$

Then a= 1 , b=-2 , c= 16 , d=4 , e= 5 , f = 11

The Addition & Subtraction of Matrices :

$A_{m\times n}$ and $B_{m\times n}$ are two matrices then $A_{m\times n}$+ $B_{m\times n}$ is $a_{ij} + b_{ij}$

And $A_{m\times n}$- $B_{m\times n}$ is $a_{ij} - b_{ij}$

Ex:

$$\begin{pmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{pmatrix} = \begin{pmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 5 & 12 \\ 9 & 4 & 8 \end{pmatrix} - \begin{pmatrix} 3 & 7 & 1 \\ 2 & 10 & -5 \end{pmatrix} = \begin{pmatrix} 6-3 & 5-7 & 12-1 \\ 9-2 & 4-10 & 8+5 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -2 & 11 \\ 7 & -6 & 13 \end{pmatrix}$$

Theorem:

Addition of matrices is commutative and associative, that is if A, B and C are matrices having the same dimension then:

A + B = B + A (commutative)

A + (b + C) = (A + B) + C (associative)

The Multiplication:

1. Scalar Multiplication :

To multiply a matrix by a single number (i.e. a scalar), each individual element of the matrix is multiplied by that factor:

$A_{m\times n}$ and K be a scalar then $K(A_{m\times n}) = (K\ a_{ij})$

EX:

A= $\begin{pmatrix} 3 & 6 \\ 0 & 5 \end{pmatrix}$ k=-2 then kA =

-2.$\begin{pmatrix} 3 & 6 \\ 0 & 5 \end{pmatrix} = \begin{pmatrix} -6 & -12 \\ 0 & -10 \end{pmatrix}$

1. Two matrices **Multiplication:**

Two matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second.

$$\text{e.g. if } \mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \text{ and } \mathbf{b} = (b_i) = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\text{then } \mathbf{A}.\mathbf{b} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{pmatrix}$$

i.e. each element in the top row of **A** is multiplied by the corresponding element in the first column of **b** and the products added. Similarly, the second row of the product is found by multiplying each element in the second row of **A** by the corresponding element in the first column of **b**.

If $\mathbf{A} = (a_{ij}) = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix}$ and $\mathbf{B} = (b_{ij}) = \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \begin{pmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{pmatrix}$

$$= \begin{pmatrix} 1\times 8+5\times 2 & 1\times 4+5\times 5 & 1\times 3+5\times 8 & 1\times 1+5\times 6 \\ 2\times 8+7\times 2 & 2\times 4+7\times 5 & 2\times 3+7\times 8 & 2\times 1+7\times 6 \\ 3\times 8+4\times 2 & 3\times 4+4\times 5 & 3\times 3+4\times 8 & 3\times 1+4\times 6 \end{pmatrix}$$

$$= \begin{pmatrix} 8+10 & 4+25 & 3+40 & 1+30 \\ 16+14 & 8+35 & 6+56 & 2+42 \\ 24+8 & 12+20 & 9+32 & 3+24 \end{pmatrix}$$

$$= \begin{pmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{pmatrix}$$

If $\mathbf{A} = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 9 & 5 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 7 & 1 \\ -2 & 9 \\ 4 & 3 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \ldots\ldots\ldots\ldots$

$$= \begin{pmatrix} 14 - 8 + 24 & 2 + 36 + 18 \\ 21 - 18 + 20 & 3 + 81 + 15 \end{pmatrix} = \begin{pmatrix} 30 & 56 \\ 23 & 99 \end{pmatrix}$$

If $\mathbf{A} = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$

$$\mathbf{A}^2 = \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 & 7 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 16 + 35 & 28 + 14 \\ 20 + 10 & 35 + 4 \end{pmatrix} = \begin{pmatrix} 51 & 42 \\ 30 & 39 \end{pmatrix}$$

Note that, in matrix multiplication, $\mathbf{A}.\mathbf{B} \neq \mathbf{B}.\mathbf{A}$, i.e. multiplication is not commutative. The order of the factors is important!

In the product $\mathbf{A}.\mathbf{B}$, $\mathbf{B}$ is *pre-multiplied* by $\mathbf{A}$
and $\mathbf{A}$ is *post-multiplied* by $\mathbf{B}$.

So, if $\mathbf{A} = \begin{pmatrix} 5 & 2 \\ 7 & 4 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 9 & 2 & 4 \\ -2 & 3 & 6 \end{pmatrix}$

then $\mathbf{A}.\mathbf{B} = \ldots\ldots\ldots\ldots$ and $\mathbf{B}.\mathbf{A} = \ldots\ldots\ldots\ldots$

$$\mathbf{A}.\mathbf{B} = \begin{pmatrix} 41 & 16 & 32 \\ 55 & 26 & 52 \\ 25 & 9 & 18 \end{pmatrix}; \quad \mathbf{B}.\mathbf{A} = \begin{pmatrix} 71 & 30 \\ 29 & 14 \end{pmatrix}$$

Types of Matrices :

1 – Row Matrix: A matrix which has exactly one row is called row matrix.

For example (1, 2, 3, 4) is row matrix

2 – Column Matrix: A matrix which has exactly one column is called a column matrix for example $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$ is a column matrix.

3 – Square Matrix: A matrix in which the number of row is equal to the number of columns is called a square matrix for example $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is a 2×2 square matrix.

A matrix (A) ($n\times n$) A is said to be order or to be an n-square matrix.

4 - Null or Zero Matrix: A matrix each of whose elements is zero is called null matrix or zero matrix, for example $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is a (2×3) null matrix.

5 – Diagonal Matrix: the elements aii are called diagonal of a square matrix (a_{11} a_{22} – a_{nn}) constitute its main diagonal A square matrix whose every element other than diagonal elements is zero is called a diagonal matrix for

Example: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

6 – Scalar Matrix: A diagonal matrix, whose diagonal elements are equal, is called a scalar matrix.

For example $\begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ are scalar matrix

7 – Identity Matrix: A diagonal matrix whose diagonal elements are all equal to 1 (unity) is called identity matrix or (unit matrix). And denoted by in for

Example $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

8 – Triangular Matrix: A square matrix (aij) whose element aij = 0 whenever $j \langle i$ is Called a lower triangular matrix.simillary y a square mat (aij) whose element aij = 0 whenever is called an upper Tringular Matrix

For example: $\begin{pmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ are lower triangular matrix

And

$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, are upper triangular

Transpose of Matrix :

If the rows and columns of a matrix are interchanged:

i.e. the first row becomes the first column,
the second row becomes the second column,
the third row becomes the third column, etc.,

then the new matrix so formed is called the *transpose* of the original matrix. If **A** is the original matrix, its transpose is denoted by $\tilde{\mathbf{A}}$ or $\mathbf{A}^T$. We shall use the latter.

EX:

If $A=\begin{pmatrix} 1 & 8 & 1.5 \\ 7 & -66 & 3.3 \end{pmatrix}$ then $A^T=\begin{pmatrix} 1 & 7 \\ 8 & -66 \\ 1.5 & 3.3 \end{pmatrix}$

9 – Symmetric Matrix: A square matrix A such that $A = A^T$ is called symmetric matrix i.e. A is a symmetric matrix if and only if aij = a_{ji} for all element.

10 – Skew symmetric Matrix: A square matrix A such that $A = A^T$ is called that A is skew symmetric matrix. i.e A is skew matrix ⟷ a_{ji} = -aij for all element of A.

The following are examples of symmetric and skew – symmetric matrices respectively

$$(a)\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix}, (b)\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$$

(a) symmetric

(b) Skew – symmetric.

Determinants of Matrix

To every a square matrix that is assigned a specific number called the determinant of matrix

the determinant of a matrix A is denoted by $|A|$

If A is (1*1) matrix, A=(a) :then $|A| = |a| = a$

If A is (2*2) matrix , A=$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$=ad-bc

1f A is (3*3) matrix ,A=$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ then

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$= $a_{11} \begin{vmatrix} a_{22} & a_{22} \\ a_{32} & a_{33} \end{vmatrix}$- $a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$ or

$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$=$[a_{11}a_{22}a_{33}$+$a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}] - [a_{12}a_{21}a_{33} + a_{11}a_{23}a_{32}$
$+ a_{13}a_{22}a_{31}]$

Note that if $|A| = 0$ then a matrix A is called a singular

the determinant of $\begin{pmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{pmatrix}$ is $\begin{vmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{vmatrix}$ and the value of this

determinant is $5(42 - 12) - 2(0 - 24) + 1(0 - 48)$

$$= 5(30) - 2(-24) + 1(-48) = 150 + 48 - 48 = 150$$

The Properties of Determinants:

(1) det A = det A^T where A^T is the transpose of A.

(2) if any two rows (or two columns) of a determinates are interchanged the value of determinants is multiplied by -1.

(3) if all elements in row (or column) of a square matrix are zero.

Then det (A) = 0

The abjoint of Matrix :

Minor of matrix : let A= $\begin{pmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \dots \dots & & \\ a_{n1} & a_{n2} \dots & a_{nn} \end{pmatrix}$ (*)

Is the square matrix of order n then the determinant of any square sub-matrix of a with order (n-1) obtained by deleting row and column is called the minor of A and denoted by Mij.

Cofactor of matrix: let A which defined in (*) with M_{ij} (minor of A) ,then the Cofactor of matrix is

$$C_{ij} = (-1)^{i+j} M_{ij}$$

ad joint of matrix : the ad joint of matrix A in(*) is the transpose of matrix Cofactor of A

i.e adj(A) = C^T

EX: let A = $\begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$ find the ad joint of A ?

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{vmatrix}$$ from which we can form a new matrix **C** of the cofactors.

$$\mathbf{C} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$ where A_{11} is the cofactor of a_{11}, A_{ij} is the cofactor of a_{ij} etc.

$$A_{11} = +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} = +(0 - 24) = -24 \qquad A_{12} = -\begin{vmatrix} 4 & 6 \\ 1 & 0 \end{vmatrix} = -(0 - 6) = 6$$

$$A_{13} = +\begin{vmatrix} 4 & 1 \\ 1 & 4 \end{vmatrix} = +(16 - 1) = 15$$

$$A_{21} = -\begin{vmatrix} 3 & 5 \\ 4 & 0 \end{vmatrix} = -(0 - 20) = 20 \qquad A_{22} = +\begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = +(0 - 5) = -5$$

$$A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} = -(8 - 3) = -5$$

$$A_{31} = +\begin{vmatrix} 3 & 5 \\ 1 & 6 \end{vmatrix} = +(18 - 5) = 13 \qquad A_{32} = -\begin{vmatrix} 2 & 5 \\ 4 & 6 \end{vmatrix} = -(12 - 20) = 8$$

$$A_{33} = +\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = +(2 - 12) = -10$$

$\therefore$ The matrix of cofactors is $$\mathbf{C} = \begin{pmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{pmatrix}$$

and the transpose of **C**, i.e. $$\mathbf{C}^{\mathrm{T}} = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

$$\text{adj}(A) = C^T = \begin{pmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{pmatrix}$$

Inverse of a square matrix :

denoted by A^{-1} is

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A) \quad , |A| \neq 0$$

EX: let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix}$ find A^{-1}?

$$|\mathbf{A}| = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{pmatrix} = 1(2-0) - 2(8-30) + 3(0-6) = 28$$

$$A_{11} = +(2-0) = 2; \quad A_{12} = -(8-30) = 22; \quad A_{13} = +(0-6) = -6$$

$$A_{21} = -(4-0) = -4; \quad A_{22} = +(2-18) = -16; \quad A_{23} = -(0-12) = 12$$

$$A_{31} = +(10-3) = 7; \quad A_{32} = -(5-12) = 7; \quad A_{33} = +(1-8) = -7$$

$$C = \begin{pmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{pmatrix}$$

And adj(A) = $C^T = \begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix}$

$$\therefore A^{-1} = \frac{1}{|A|} \text{adj(A)}$$

$$= \frac{1}{28}\begin{pmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{pmatrix} = \begin{pmatrix} \frac{2}{28} & \frac{-4}{28} & \frac{7}{28} \\ \frac{22}{28} & \frac{-16}{28} & \frac{7}{28} \\ \frac{-6}{28} & \frac{12}{28} & \frac{-7}{28} \end{pmatrix}$$

<u>The properties of multiplication</u>

1 – (KA) B = K (AB) = A (KB)   K is any number

2 – A (BC) = (AB) C

3 – (A + B) C = AC + BC

4 – C (A + B) = CA + CB

5 – AB ‡ BA (in general)

For example: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$B\,A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

A B = ‡ B A

6 – A B = 0 but not necessarily A = 0 or B = 0

For Example: $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, B = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}$

$$A\,B = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$A \neq 0, B \neq 0$$

$9 - (A\,B)^T = B^T A^T$

$10 - A^{-1} A = A.A^{-1} = I$

Solution the System of Linear Equations;

Consider the set of linear equations:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots\dots\dots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots\dots\dots + a_{2n}x_n &= b_2 \\
\vdots \qquad\quad \vdots \qquad\qquad\qquad\qquad\quad \vdots \quad &\quad \vdots \\
a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots\dots\dots + a_{nn}x_n &= b_n
\end{aligned}
$$

From our knowledge of matrix multiplication, this can be written in matrix form:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \qquad \text{i.e. } \mathbf{A.x} = \mathbf{b}$$

$$\text{where } \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}; \quad \text{and } \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

If we multiply both sides of the matrix equation by the inverse of **A**, we have:

$$\mathbf{A}^{-1}.\mathbf{A}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$$

But $\mathbf{A}^{-1}.\mathbf{A} = \mathbf{I} \quad \therefore \; \mathbf{I}.\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} \quad$ i.e. $\mathbf{x} = \mathbf{A}^{-1}.\mathbf{b}$

EX; solve the following system:

$$x_1 + 2x_2 + x_3 = 4$$

$$3x_1 - 4x_2 - 2x_3 = 2$$

$$5x_1 + 3x_2 + 5x_3 = -1$$

First write the set of equations in matrix form, which gives …

$$A=\begin{pmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 5 & 3 & 5 \end{pmatrix} \quad x=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad b=\begin{bmatrix} 4 \\ 2 \\ -1 \end{bmatrix}$$

x= A^{-1}.b

we find A^{-1}

$$\mathbf{A}^{-1} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix}$$

$$\therefore\ \mathbf{x} = \mathbf{A}^{-1}.\mathbf{b} = -\frac{1}{35}\begin{pmatrix} -14 & -7 & 0 \\ -25 & 0 & 5 \\ 29 & 7 & -10 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 2 \\ -1 \end{pmatrix} = \ldots\ldots\ldots\ldots \qquad \mathbf{x} = -\frac{1}{35}\begin{pmatrix} -70 \\ -105 \\ 140 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$$

So finally $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix} \quad \therefore\ x_1 = 2;\ \ x_2 = 3;\ \ x_3 = -4$

Grammer's Rule :

Consider the set of linear equations:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots\ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots\ldots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad\qquad\qquad \vdots \quad &\quad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots\ldots + a_{nn}x_n &= b_n \end{aligned}$$

If D $=\begin{vmatrix} a_{11} & a_{12}\ldots\ldots & a_{1n} \\ a_{21}\ldots\ldots & a_{22}\ldots\ldots & a_{2n} \\ a_{n1} & a_{n2}\ldots\ldots & a_{nn} \end{vmatrix} \neq 0$

Then the system has a unique solution by **<u>Grammer's Rule is</u>**

$$x_1 = \frac{1}{D}\begin{vmatrix} b_1 & a_{12}\ldots\ldots & a_{n1} \\ b_2 & a_{22}\ldots\ldots & a_{n2} \\ \vdots & \vdots & \\ b_n & a_{2n}\ldots\ldots & a_{nn} \end{vmatrix} = \frac{Dx_1}{D} \quad ,, x_2 = \frac{1}{D}\begin{vmatrix} a_{11} & b_1\ldots\ldots & a_{n1} \\ a_{21} & b_2\ldots\ldots & a_{n2} \\ \vdots & \vdots & \\ a_{n1} & b_n\ldots\ldots & a_{nn} \end{vmatrix} = \frac{Dx_2}{D}\ldots\ldots\ldots$$

$$x_n = \frac{1}{D}\begin{vmatrix} a_{11} & a_{12}\ldots\ldots & b_1 \\ a_{21}\ldots\ldots & a_{22}\ldots\ldots & b_2 \\ & & \vdots \\ a_{n1} & a_{n2}\ldots\ldots & b_n \end{vmatrix} = \frac{Dx_n}{D}$$

EX: solve the system :

$$3x_1 - x_2 = 9$$

$$x_1 + 2x_2 = -4$$

So, the system can put in the form

$$\begin{pmatrix} 3 & -1 \\ 1 & 2 \end{pmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{pmatrix} 9 \\ -4 \end{pmatrix}$$

$$D = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = 6+1=7$$

$$x_1 = \frac{1}{7}\begin{vmatrix} 9 & -1 \\ -4 & 2 \end{vmatrix} = \frac{1}{7}(18-4) = \frac{14}{7} = 2$$

$$x_2 = \frac{1}{7}\begin{vmatrix} 3 & 9 \\ 1 & -4 \end{vmatrix} = \frac{1}{7}(-12-9) = \frac{-21}{7} = -3$$

EX: solve the system :

$$X_1 + 3X_2 - 2X_3 = 11$$
$$4X_1 - 2X_2 + X_3 = -15$$
$$3X_1 + 4X_2 - X_3 = 3$$

The system (1) become $\begin{pmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{pmatrix}\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{pmatrix} 11 \\ -15 \\ 3 \end{pmatrix}$

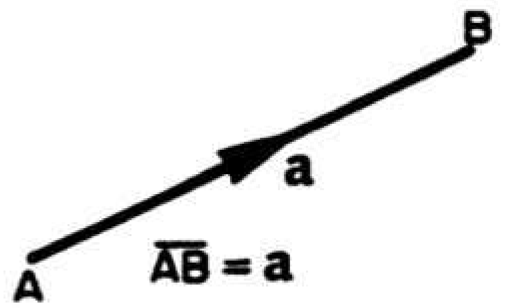
Since $D = \det = \begin{vmatrix} 1 & 3 & -2 \\ 4 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix} = -25$

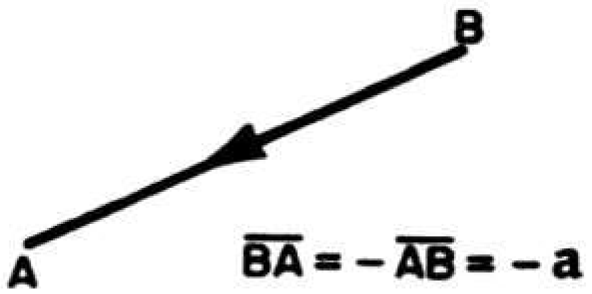
$$X_1 = \frac{\begin{vmatrix} 11 & 3 & -2 \\ -15 & -2 & 1 \\ 3 & 4 & -1 \end{vmatrix}}{-25} = \frac{50}{-25} = -2$$

$$X_2 = \frac{\begin{vmatrix} 1 & 11 & -2 \\ 4 & -15 & 1 \\ 3 & 3 & -1 \end{vmatrix}}{-25} = \frac{-25}{-25} = 1 \qquad X_3 = \frac{\begin{vmatrix} 1 & 3 & 11 \\ 4 & -2 & -15 \\ 3 & 4 & 3 \end{vmatrix}}{-25} = \frac{125}{-25} = -3$$

Vectors:

A vector in the plane is a directed line segment . A directed line segment $\overrightarrow{AB}$ and the length is denoted by $\left|\overrightarrow{AB}\right|$





Two equal vectors :

If two vectors, **a** and **b**, are said to be equal, they have the same magnitude and the same direction.

If **a** = **b**, then

(a) $a = b$ (magnitudes equal)

(b) the direction of **a** = direction of **b**, i.e. the two vectors are parallel and in the same sense.

Vector from two points :

Let $p_1 = (x_1, y_1)$ and $p_2 = (x_2, y_2)$ then the vector $\overrightarrow{p_1p_2}$ is

$\overrightarrow{p_1p_2} = (x_2 - x_1, y_2 - y_1)$

And the length of $|\overrightarrow{p_1p_2}|$ is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ (magnitude of vector)

EX: let A= (2,3) and B=(1,-2) , find $\overrightarrow{AB}$.

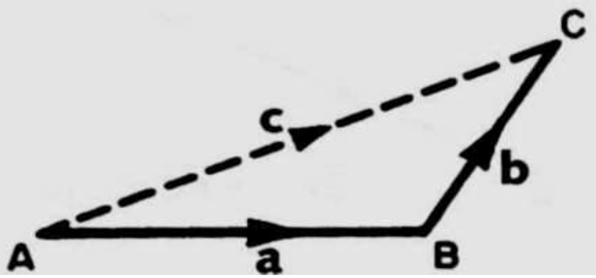
$$\overrightarrow{AB} = (1-2, -2-3) = (-1, -5) and$$

$$|\overrightarrow{AB}| = \sqrt{(1-2)^2 + (-2-3)^2} = \sqrt{1+25} = \sqrt{26}$$

Note: in two dimension then the components of a vector a is (a_1,a_2) and in three dimension is (a_1,a_2, a_3)

The Addition of Two Vectors :The Addition of Two Vectors :

The sum of two vectors, $\overline{AB}$ and $\overline{BC}$, is defined as the single or equivalent or resultant vector $\overline{AC}$



i.e. $\overline{AB} + \overline{BC} = \overline{AC}$

or $\mathbf{a} + \mathbf{b} = \mathbf{c}$

Theorem:

Let a=$(a_1, a_2), b = (b_1, b_2) in\ R^2, and\ K\ be\ a\ scalar\ .then$

a. a+ b = $(a_1 + b_1, a_2 + b_2)$ b- ka =(Ka_1, Ka_2)

and if a=$(a_1, a_2, a_3), b = (b_1, b_2, b_3) in\ R^3, and\ K\ be\ a\ scalar\ .then$

a. a+ b = $(a_1 + b_1, a_2 + b_2, a_3 + b_3)$ b- ka =$(Ka_1, Ka_2, Ka_3$

EX: $v = (4, -1), w = (3,2)$ be are vectors find $v + w$

$$v + w = (4 + 3, -1 + 2) = (7,1)$$

Theorem

For any vectors a,b,c and scalars k ,l, we have

(a) a+ b =b +a Commutative Law

(b) a+(b+c) = (a +b)+c Associative Law

(c) a+0=0+a Additive Identity

(d) a+(-a)=0 Additive Inverse

(e) k(la) = (kl)a Associative Law

(f) k(a+b) = ka+ kb Distributive Law

(g) (k + l)a = ka+ l b Distributive Law

Example Let v = (2,1,−1) and w = (3,−4,2) in R^3.

(a) Find v−w.

Solution: v−w = (2−3,1−(−4),−1−2) = (−1,5,−3)

(b) Find 3v+2w.

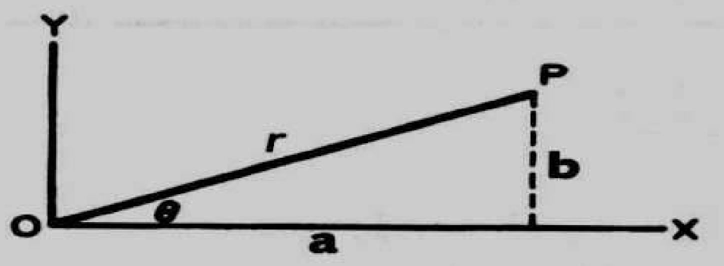
Solution: 3v+2w = (6,3,−3)+(6,−8,4) = (12,−5,1)

The unit vectors:

A vector with length equal 1 is called unit vector .The standard unit vectors is

i= (1,0) ,j= (0,1) in two dimension and in 3D is i=(1,0,0) ,j=(0,1,0) ,k=(0,0,1)

Component of Vector in Term of Unit Vectors:



The vector $\overline{OP}$ is defined by its magnitude (r) and its direction (θ). It could also be defined by its two components in the OX and OY directions.

i.e. $\overline{OP}$ is equivalent to a vector **a** in the OX direction + a vector **b** in the OY direction.

i.e. $\overline{OP} = \mathbf{a}$ (along OX) $+ \mathbf{b}$ (along OY)

If we now define **i** to be a *unit vector* in the OX direction,

then $\mathbf{a} = a\,\mathbf{i}$

Similarly, if we define **j** to be a *unit vector* in the OY direction,

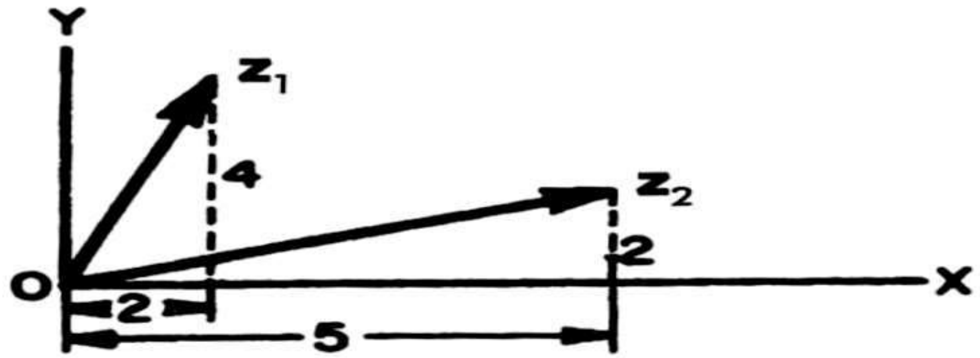
then $\mathbf{b} = b\,\mathbf{j}$

So that the vector OP can be written as:

$\mathbf{r} = a\mathbf{i} + b\,\mathbf{j}$

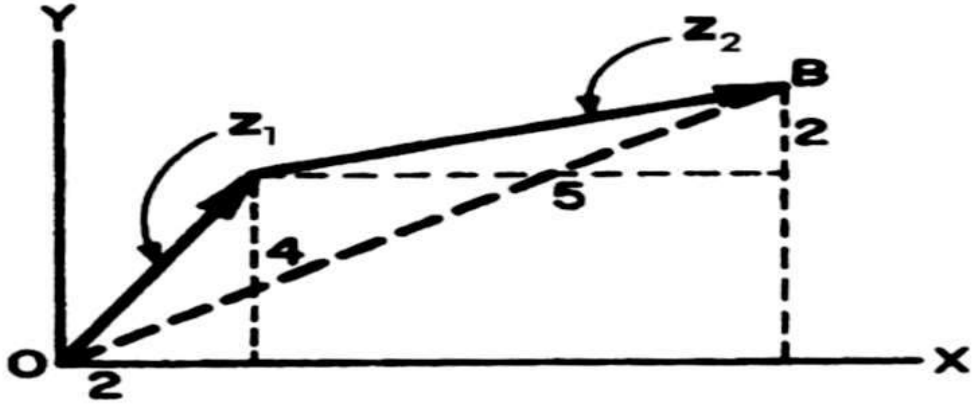
where **i** and **j** are unit vectors in the OX and OY directions.

Let $\mathbf{z}_1 = 2\mathbf{i} + 4\mathbf{j}$ and $\mathbf{z}_2 = 5\mathbf{i} + 2\mathbf{j}$

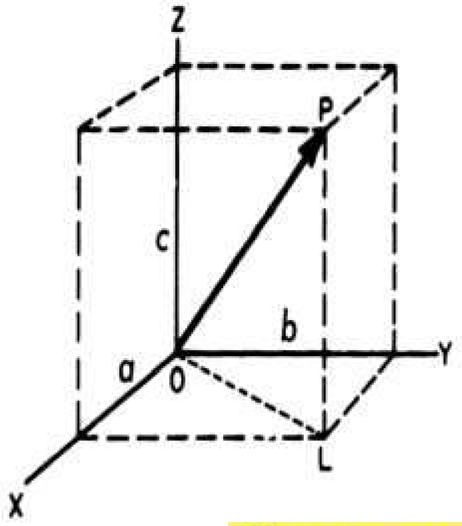


To find $\mathbf{z}_1 + \mathbf{z}_2$, draw the two vectors in a chain.

$$\mathbf{z}_1 + \mathbf{z}_2 = \overline{OB} = (2+5)\mathbf{i} + (4+2)\mathbf{j} = 7\mathbf{i} + 6\mathbf{j}$$



Vector in The Space :



Vector $\overline{OP}$ is defined by its components

a along OX

b along OY

c along OZ

Let $\mathbf{i}$ = unit vector in OX direction

$\mathbf{j}$ = unit vector in OY direction

$\mathbf{k}$ = unit vector in OZ direction

Then $$\overline{OP} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

Also $$OL^2 = a^2 + b^2 \text{ and } OP^2 = OL^2 + c^2$$

$$OP^2 = a^2 + b^2 + c^2$$

So, if $\mathbf{r} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then $r = \sqrt{a^2 + b^2 + c^2}$

This gives us an easy way of finding the magnitude of a vector expressed in terms of the unit vectors.

EX:

a= -i+3j+k   and b= 4i+7j .find

1. 2a+3b

2(-i+3j+k) +3(4i+7j) = (-2i+6j+2k)+(12i+21j) = 10i+27j+2k

1. a-b

(-i+3j+k) –(4i+7j)= -5i- 4j+k

2. $\left|\frac{1}{2}a\right|$

$$=\sqrt{(-\frac{1}{2})^2+(\frac{3}{2})^2+(\frac{1}{2})^2}=\sqrt{\frac{1}{4}+\frac{9}{4}+\frac{1}{4}}=\sqrt{\frac{11}{4}}=\frac{\sqrt{11}}{2}$$

The Dot Product:

Let a= (a_1, a_2, a_3) $and\ b = (b_1, b_2, b_3)$ in 3D then, the dot product denoted by a.b is $a_1b_1 + a_2b_2 + a_3b_3$

Similarly if a= $a_1i + a_2j + a_3k$ $and\ b = b_1i + b_2j + b_3k$

a.b= $a_1b_1 + a_2b_2 + a_3b_3$

EX:

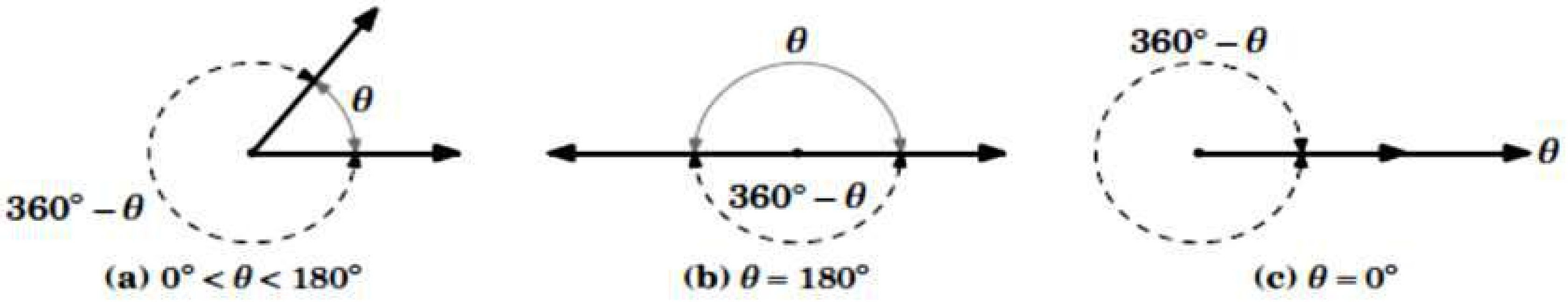
If p=3i-2j+k and q=2i+3j-4k

$$\begin{aligned}\mathbf{p}.\mathbf{q} &= 3 \times 2 + (-2) \times 3 + 1 \times (-4) \\ &= 6 - 6 - 4 \\ &= -4\end{aligned}$$

$$\therefore\ \mathbf{p}.\mathbf{q} = -4$$

The Angle between Two Vectors :

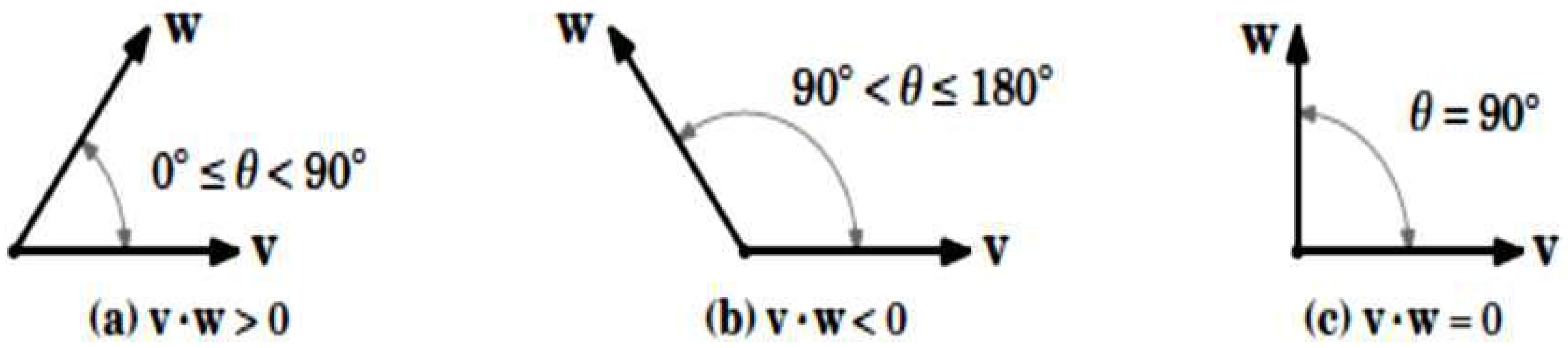
The angle between two non zeros vectors is the smallest angle between them .



Theorem:

Let v ,w are two non zeros vectors and θ *is*the angle between a and b then

$$\cos\theta = \frac{v.w}{\|v\|\|w\|}$$



EX:

Is a= -i+5j+2k perpendicular to b=3i+j-k ?

a.b = (-1.3+5.1+2.-1)=0 a is perpendicular to b

Theorem :

For any vectors u,v,w and k be a scalar :

(a) $\mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$	Commutative Law
(b) $(k\mathbf{v})\cdot\mathbf{w}=\mathbf{v}\cdot(k\mathbf{w})=k(\mathbf{v}\cdot\mathbf{w})$	Associative Law
(c) $\mathbf{v}\cdot\mathbf{0}=0=\mathbf{0}\cdot\mathbf{v}$	
(d) $\mathbf{u}\cdot(\mathbf{v}+\mathbf{w})=\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$	Distributive Law
(e) $(\mathbf{u}+\mathbf{v})\cdot\mathbf{w}=\mathbf{u}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{w}$	Distributive Law
(f) $\lvert\mathbf{v}\cdot\mathbf{w}\rvert\le\lVert\mathbf{v}\rVert\,\lVert\mathbf{w}\rVert$	Cauchy-Schwarz Inequality[5]

The cross product:

Let a= $a_1i + a_2j + a_3k$ *and* $b = b_1i + b_2j + b_3k$ in 3D then, the cross product denoted by a×b is *the vector* :

$$a\times b=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k=$$

$$(a_2 b_3-a_3 b_2) i-(a_1 b_3-a_3 b_1) j+(a_1 b_2-a_2 b_1) k$$

EX:

let p=3i-4j+2k and q= 2i+5j-k ,find p×q?

$$\mathbf{p}\times\mathbf{q}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 2 \\ 2 & 5 & -1 \end{vmatrix}$$

$$=\mathbf{i}\begin{vmatrix} -4 & 2 \\ 5 & -1 \end{vmatrix}-\mathbf{j}\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix}+\mathbf{k}\begin{vmatrix} 3 & -4 \\ 2 & 5 \end{vmatrix}$$

$$=\mathbf{i}(4-10)-\mathbf{j}(-3-4)+\mathbf{k}(15+8)$$

$$=-6\mathbf{i}+7\mathbf{j}+23\mathbf{k}$$

Theorem 1.14. For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^3$, and scalar k, we have

(a) $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$ — Anticommutative Law

(b) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ — Distributive Law

(c) $(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = \mathbf{u} \times \mathbf{w} + \mathbf{v} \times \mathbf{w}$ — Distributive Law

(d) $(k\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (k\mathbf{w}) = k(\mathbf{v} \times \mathbf{w})$ — Associative Law

(e) $\mathbf{v} \times \mathbf{0} = \mathbf{0} = \mathbf{0} \times \mathbf{v}$

(f) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$

(g) $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v} \parallel \mathbf{w}$

If θ is the angle between nonzero vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$, then

$$\|\mathbf{v} \times \mathbf{w}\| = \|\mathbf{v}\| \, \|\mathbf{w}\| \sin\theta$$

Function

Numbers:

1. N={1,2,3,…….} the set of Natural numbers

2-Z={-1,-2,…..,0,1,2,……} the set of integer numbers

3-Q={ N,Z, $\frac{p}{q}$, q$\neq$ 0 and p,q ϵZ } the set of rational numbers

4- H= $\{\sqrt{x}\}$ the set of irrational numbers

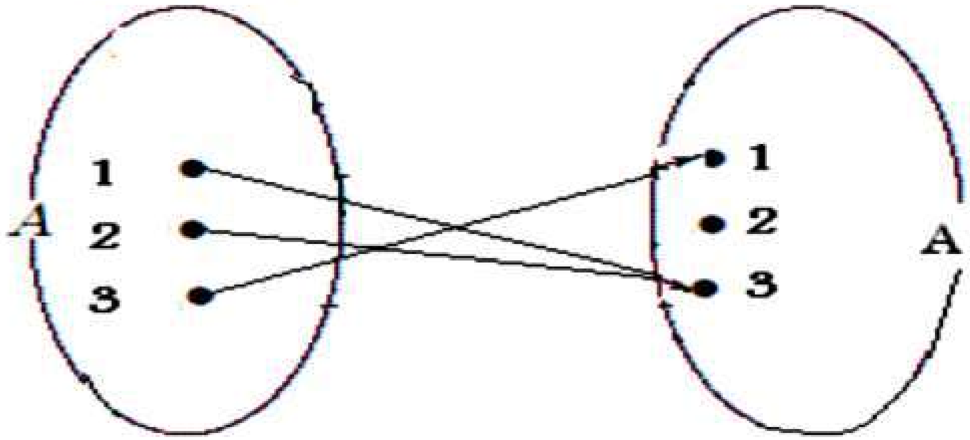
5- R=Q+H = the set of real numbers

Interval

The set of values that a variable χ may take on is called the domain of χ. The domains of the variables in many applications of calculus are intervals like those shows below.

Let A,B be two nonempty sets, a function F: A→B is a rule which associates with **each** element of A a **unique** element in B.
The set A is called the **domain** of the function, and the set B is called the **range** of the function.

consider the following relation on the set A={1,2,3}
F = {(1,3),(2,3),(3,1)}
F is a function



Ex: if f(x)=x^3 then f(2) =$2^3 = 8$

Ex: verify the domains and ranges of these functions :

y= x^2

domain(x)= (-∞, ∞) range(y)=[0,∞)

y=$1/x$

domain(x)= (-∞, 0) ∪ (0, ∞) range(y)= (-∞, 0) ∪ (0, ∞)

y=$\sqrt{x}$

domain(x)= [0, ∞) range(y)= [0, ∞)

y= $\sqrt{4-x}$

domain(x)= $(-\infty, 4)$ range(y)= [0, ∞)

The gargh of function

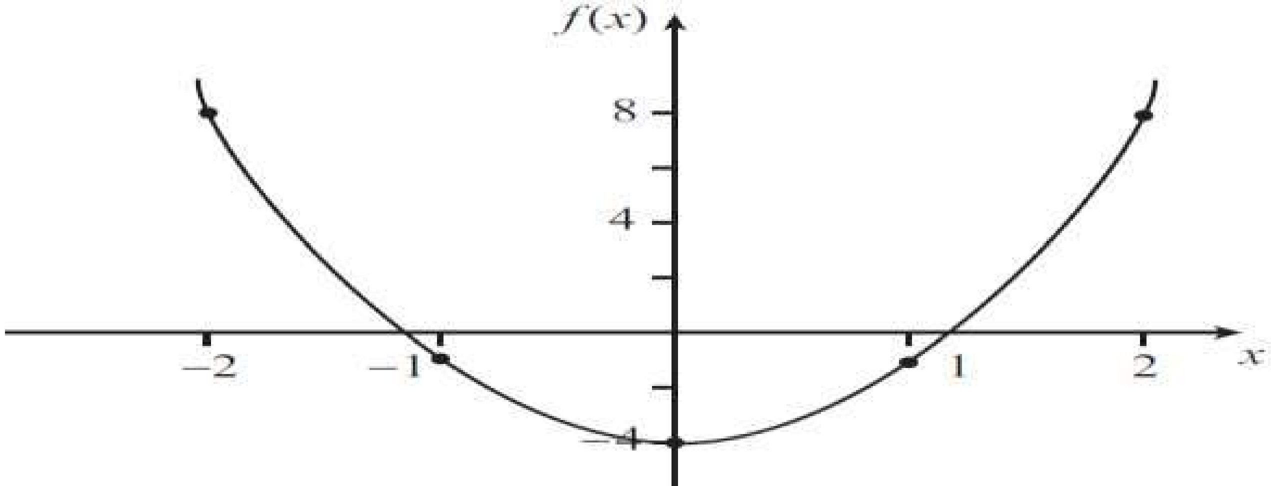
If we have a function given by a formula, we can try to plot its graph. Suppose, for example, that we have a function f defined by

$$f(x) = 3x^2 - 4.$$

The argument of the function (the independent variable) is x, and the output (the dependent variable) is $3x^2 - 4$. So we can calculate the output of the function for different arguments:

$$\begin{aligned} f(0) &= 3 \times 0^2 - 4 &&= -4 \\ f(1) &= 3 \times 1^2 - 4 &&= -1 \\ f(2) &= 3 \times 2^2 - 4 &&= 8 \\ f(-1) &= 3 \times (-1)^2 - 4 &&= -1 \\ f(-2) &= 3 \times (-2)^2 - 4 &&= 8. \end{aligned}$$

We can put this information into a table to help us plot the graph of the function.



Ex:

x

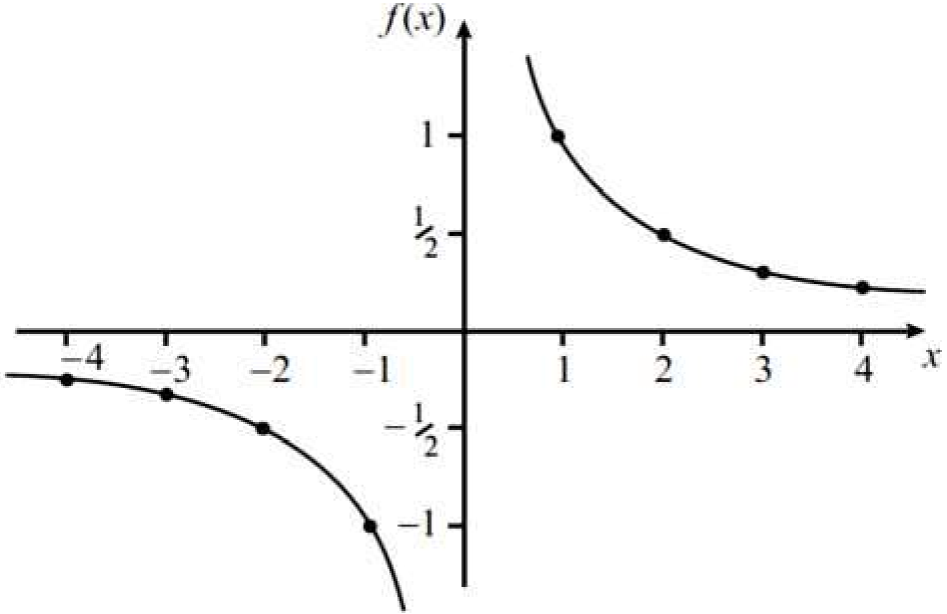
As usual, the first step is to check some values.

$$f(1) = 1, \qquad f(2) = \tfrac{1}{2}, \qquad f(3) = \tfrac{1}{3}, \qquad f(4) = \tfrac{1}{4},$$

$$f(-1) = \frac{1}{(-1)} = -1, \qquad f(-2) = \frac{1}{(-2)} = -\tfrac{1}{2},$$

$$f(-3) = \frac{1}{(-3)} = -\tfrac{1}{3}, \qquad f(-4) = \frac{1}{(-4)} = -\tfrac{1}{4}.$$

When we try to calculate $f(0)$ we have a problem, because we cannot divide by zero. So we have to restrict the domain to exclude $x = 0$.



Note 2: Let $f(\chi)$ and $g(\chi)$ be two function.

1 - $(f \pm g)(\chi) = f(\chi) \pm g(\chi)$

2 - $(f.g)(\chi) = f(\chi) \,.\, g(\chi)$

3 - $(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)}$ if g $(\chi) \neq 0$

Example: Let $f(\chi) = \chi + 2, g(\chi) = \sqrt{\chi - 3}$ evaluate

$f \pm g \,,\, f.g \; and \; \frac{f}{g}$

So: $(f \pm g)(\chi) = f(\chi) \pm g(\chi) = \chi + 2 \pm (\sqrt{\chi - 3}$

$(f.g)(\chi) = f(\chi) \,.\, g(\chi) = (\chi + 2)(\sqrt{\chi - 3}$

$(\frac{f}{g})(\chi) = \frac{f(\chi)}{g(\chi)} = \frac{\chi + 2}{\sqrt{\chi - 3}}$ $\{X : X \rangle 3\}$

Composition of Function:

Let $f(\chi)$ and $g(\chi)$ be two functions

We define: $(fog)(\chi) = f(g(\chi))$

Example: Let $f(\chi) = \chi^2$, $g(\chi) = \chi - 7$ evaluate fog and gof

So: $(fog)(\chi) = f[g(x)] = f(\chi - 7) = (\chi - 7)^2$

$(gof)(\chi) = g[f(\chi)] = g(\chi^2) = \chi^2 - 7$

$\therefore fog \neq gof$

Inverse Function

Given a function F with domain A and the range B.

The inverse function of f written f, is a function with domain B and range

A such that for every $y \in B$ there exists only $\chi \in A$ with $\chi = f^{-1}(y)$.

Note that: $f^{-1} \neq \frac{1}{f}$

Even Function:

F (χ) is even if f (-x) = F (x)

Example: 1 - F $(\chi) = (\chi)^2$ is even since $f(-\chi) = (-\chi)^2 = (\chi)^2 = f(\chi)$

2 - F $(\chi) = \cos(\chi)$ is even because $f(-\chi) = \cos(-\chi) = \cos(\chi) = f(\chi)$

Odd Function:

If $f(-\chi) = -f(\chi)$ the function is called odd.

Example: 1 - $f(\chi) = \chi^3$ is odd since $f(-\chi) = -\chi 3 = -f(\chi)$

2 - $f(\chi) = Sin(-\chi) = -Sin\,X = -f(\chi)$.

Trigonometric function :

1. $\sin\theta = \frac{a}{c}$

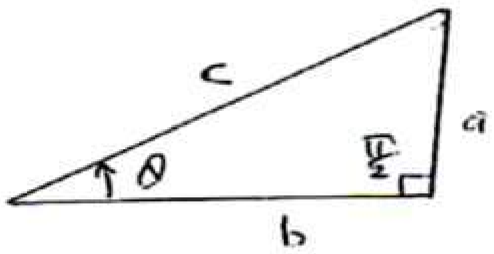
$2 - \text{Cos} \quad \varphi = \frac{b}{c}$

$3 - \tan \quad \varphi = \frac{a}{b}$

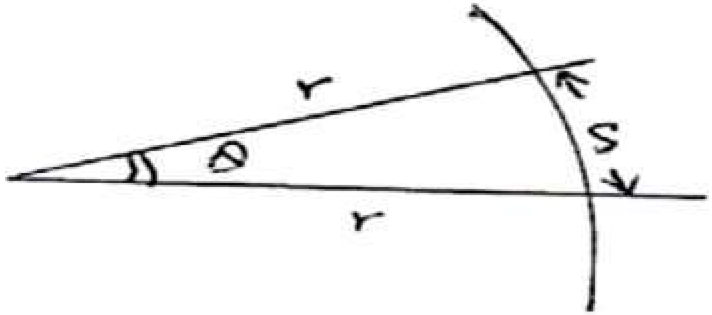
$4 - \text{Cotan} \ \varphi = \frac{1}{\tan\varphi} = \frac{b}{a}$

$5 - \text{Sec} \quad \varphi = \frac{1}{\cos\varphi} = \frac{c}{b}$

$6\text{- CSC} \quad \varphi = \frac{1}{Sin\,\varphi} = \frac{c}{a}$



Relation ships between degrees and radians



φ In radius $= \frac{s}{r}$

$$360^{\circ} = \frac{2\pi r}{r}$$

$$= 2\pi \, radius$$

$$1^{\circ} = \frac{\pi}{180} \text{ radius} = 0.0174 \text{ radian}$$

$$1 \text{ radian} = \frac{180}{\pi} \deg ree = 57.29578^{\circ}$$

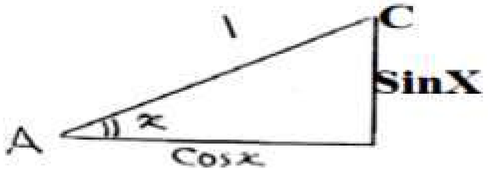
$$\left(\frac{360}{2\pi}\right) = 1 radian = 57^{\circ}.18$$

$$180^{\circ} = \pi \text{ radians} = 3.14159 - \text{radians}$$

$$1^{\circ} = \frac{2\pi}{360} = \frac{\pi}{180} \approx 0.001754 \text{ radians}$$

$$\tan\chi = \frac{\sin\chi}{\cos\chi}$$

$$Cot\chi = \frac{Cos\chi}{Sin\chi} = \frac{1}{\tan\chi}$$



$$Sec\,\chi = \frac{1}{\cos\chi}$$

$$Csc\ \chi = \frac{1}{\sin\chi}$$

$$Cos^2\chi + Sin^2\chi = 1$$

$$\tan^2\chi + 1 = Sec^2\chi$$

$$Cot^2\chi + 1 = Csc^2 X$$

$$Sin(\chi \pm y) = Sin \times Cos y\ \mu\ Cos \times Sin\,y$$

$$Cos(\chi \pm y) = Cos \times Cos y \pm Sin \times Sin y$$

$$\tan(\chi \pm y) = \frac{\tan x \pm \tan y}{1\ \mu \tan x \tan y}$$

$$1 - \quad SinA + Sin\,B = 2\,Sin\,\frac{A+B}{2}\,Cos\,\frac{A-B}{2}$$

$$2 - \ Sin\ A - Sin\ B = 2 Cos\ \frac{A+B}{2}\ Sin\ \frac{A-B}{2}$$

$$3 - \ Cos\ A + Cos\ b = 2 Cos\ \frac{A+B}{2}\ Cos\ \frac{A-B}{2}$$

$$4 - \ Cos\ A - Cos\ B = 2 Sin \frac{A+B}{2}\ Sin \frac{A-B}{2}$$

$$Sin\ 2X = 2\ Sin\ X\ CosX$$

$$Cos^{2} = Cos^{2} X - Sin^{2} X$$

$$= 1 - 2\ Sin^{2} X$$

$$= 2 Cos^{2} X - 1$$

$$Cos^{2} x = \frac{1 + Cos^{2} x}{2}$$

$$Sin^{2}\ x = \frac{1 - Cos^{2} x}{2}$$

$$Sin(\varphi + 2\pi) = Sin\varphi$$

$$Cos(\varphi + 2\pi) = Cos\varphi$$

$$\tan(\varphi + \pi) = \tan\varphi$$

Degree	O°	30°	45°	60°	90°	180°	270°	360°
θ radius	O	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sin θ	O	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\sqrt{\frac{3}{2}}$	1	O	-1	O
Cos θ	1	$\sqrt{\frac{3}{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	O	-1	O	1
tan θ	O	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$				

$$Cos(\varphi + 2n\pi) = Cos\,\varphi$$

$$Sin\ (\varphi + 2n\pi) = Sin\,\varphi$$

$$Cos(-\varphi) = Cos\,\varphi$$

$$Sin\ (-\varphi) = -Sin\,\varphi$$

$$Cos(\frac{\pi}{2} + \varphi) = Cos\,\varphi$$

$$\tan\ (\pi - \varphi) = -\tan\varphi$$

$$\tan\ (\frac{\pi}{2} + \varphi) = -Cot\varphi$$

Y=Sin x ,y= cos x, y= tan x, y=cot x , y= sec x, y=csc x,

1. y=sin x : domain (x) $-\infty \le x \le \infty$ range(y) -1$\le y \le 1$
2. y= cos x : domain (x) $-\infty \le x \le \infty$ range(y) -1$\le y \le 1$
3. y= tan x : domain (x)={x$\neq \pi/2, x \neq 3\pi/2$}

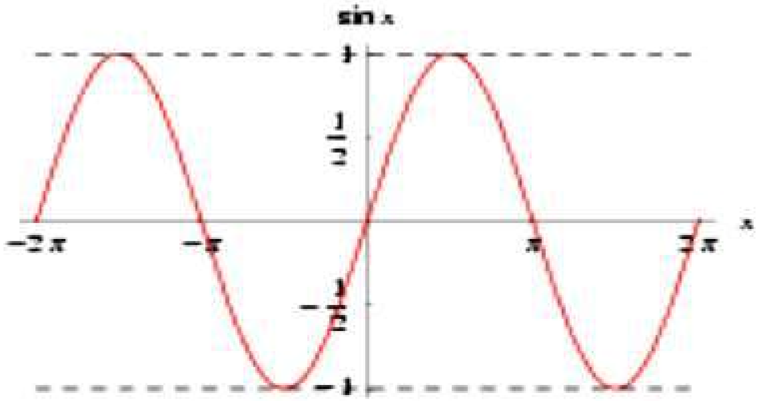


Fig.4a. Graph of sin(x).

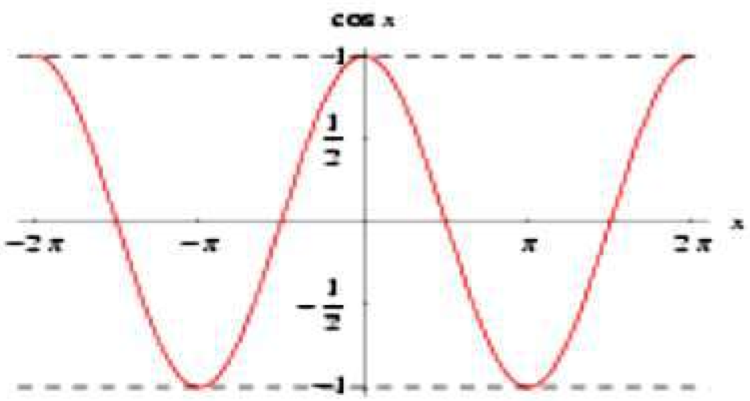


Fig.4b. Graph of cos(x).

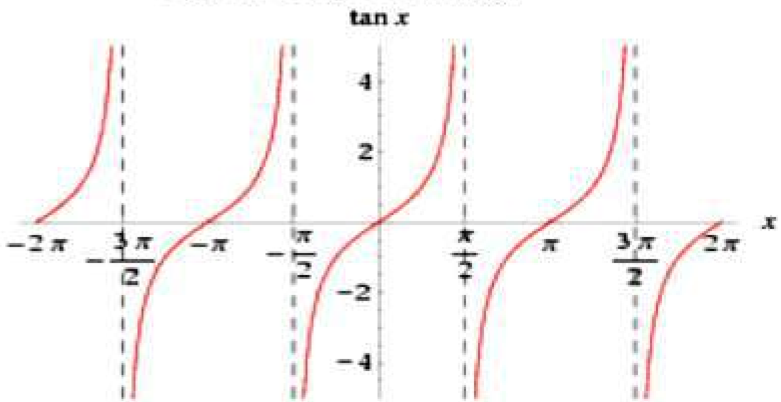


Fig.4c. Graph of tan(x).

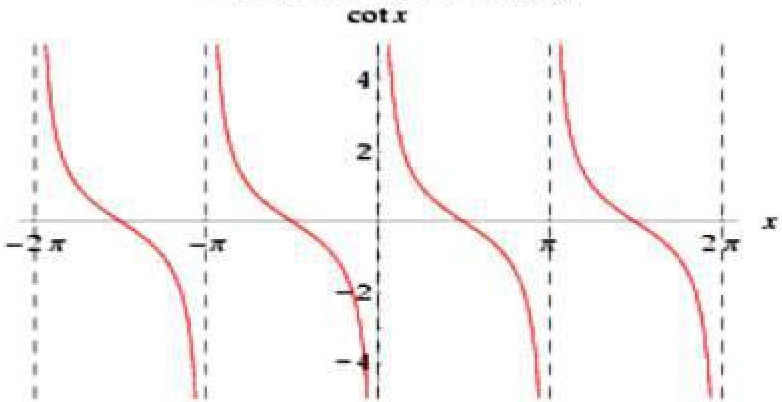


Fig.4d. Graph of cot(x).

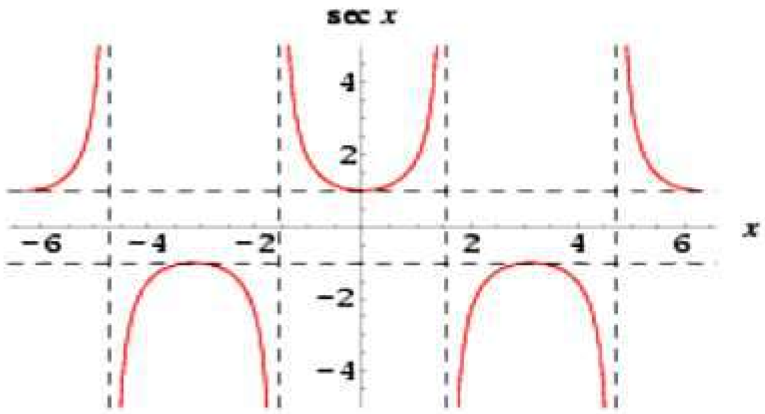


Fig.4e. Graph of sec(x).

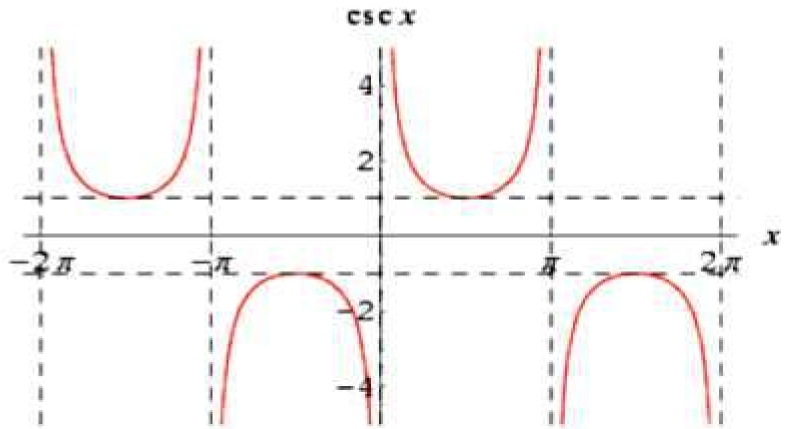


Fig.4f. Graph of csc(x).

Absolute Value:

We define the absolute value function $y = |\chi|$, the function assign every negative number to non-negative, which corresponding points.

The absolute values of X:

$$|X| = \sqrt{X^2} = \begin{cases} \chi & if X \geq 0 \\ -\chi & if X \langle 0 \end{cases}$$

Then:

1 - $|a.b| = |a|,|b|$

2 - $|a + b| \leq |a| + |b|$

3 - $|a| \leq C \Leftrightarrow -C \leq a \leq C$

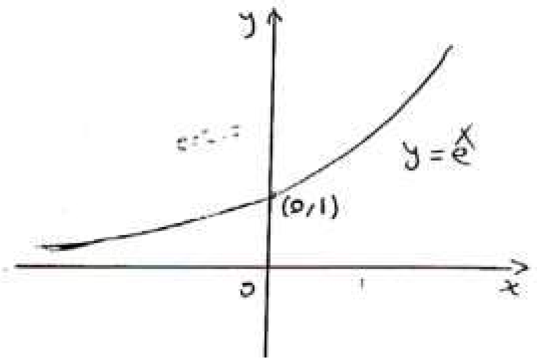
Special Functions:

1 - Exponential function

(i) $y = e^{x}$, $e = 2.7$

Domain: R

Range: R $(0, \infty)$



(ii) $y = e^{-x}$

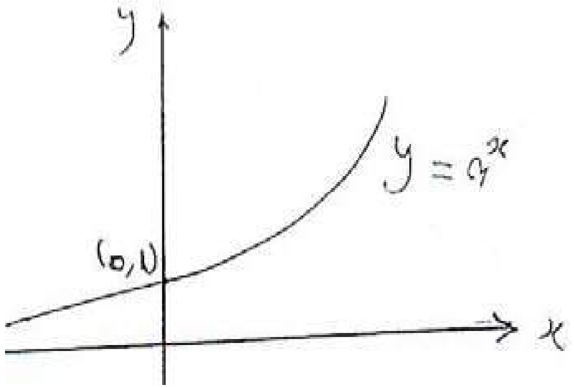
Domain: R

Range: R: $(0, \infty)$

(iii) $y = a^x$, $a \rangle 0$

Domain: R

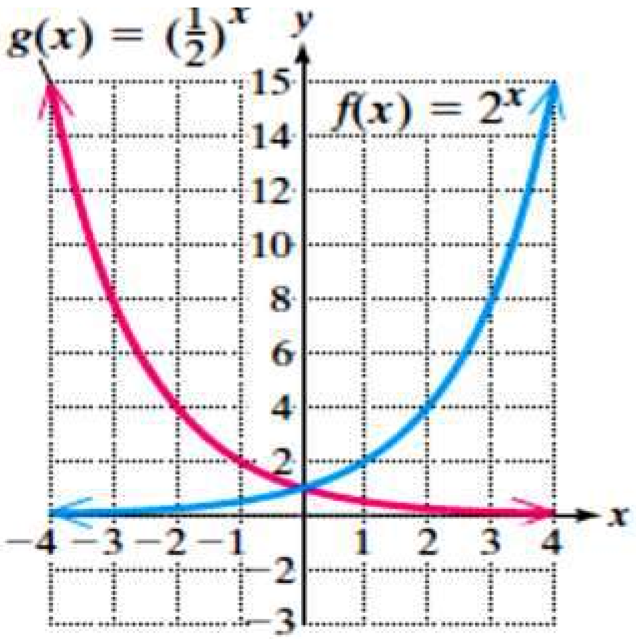
Range: $(0, \infty)$



Graph the functions *f* and *g*.

a. $f(x) = 2^x$ **b.** $g(x) = (\frac{1}{2})^x$

x	$f(x) = 2^x$	$g(x) = (\frac{1}{2})^x$
−4	$\frac{1}{16}$	16
−3	$\frac{1}{8}$	8
−2	$\frac{1}{4}$	4
−1	$\frac{1}{2}$	2
0	1	1
1	2	$\frac{1}{2}$
2	4	$\frac{1}{4}$
3	8	$\frac{1}{8}$
4	16	$\frac{1}{16}$



2 – Logarithmic Function:

(i) Common logarithmic function (Log X)

$$y = Log_{10} X \Leftrightarrow \chi = 10^{Y}$$

Domain: $(0, \infty)$

Range: R

Properties of Logarithms

Let b, x, and y be positive real numbers where $b \neq 1$, and let p be a real number. Then the following properties of logarithms are true.

1. $\log_b 1 = 0$

2. $\log_b b = 1$

3. $\log_b b^p = p$

4. $b^{\log_b x} = x$

5. $\log_b (xy) = \log_b x + \log_b y$ — **Product property for logarithms**

6. $\log_b \left(\frac{x}{y}\right) = \log_b x - \log_b y$ — **Quotient property for logarithms**

7. $\log_b x^p = p \log_b x$ — **Power property for logarithms**

Write the expressions as the sum or difference of logarithms of x, y, and z. Assume all variables represent positive real numbers.

a. $\log_3\left(\frac{xy^3}{z^2}\right)$ **b.** $\log\left(\frac{\sqrt{x+y}}{10}\right)$ **c.** $\log_b\sqrt[5]{\frac{x^4}{yz^3}}$

Solution:

a. $\log_3\left(\frac{xy^3}{z^2}\right)$

$= \log_3 xy^3 - \log_3 z^2$	Quotient property for logarithms (property 6)
$= [\log_3 x + \log_3 y^3] - \log_3 z^2$	Product property for logarithms (property 5)
$= \log_3 x + 3\log_3 y - 2\log_3 z$	Power property for logarithms (property 7)

b. $\log\left(\frac{\sqrt{x+y}}{10}\right)$

$= \log(\sqrt{x+y}) - \log(10)$ — Quotient property for logarithms (property 6)

$= \log(x+y)^{1/2} - 1$ — Write $\sqrt{x+y}$ as $(x+y)^{1/2}$ and simplify $\log 10 = 1$.

$= \frac{1}{2}\log(x+y) - 1$ — Power property for logarithms (property 7)

$= \frac{1}{5}[4\log_b x - \log_b y - 3\log_b z]$ — Power property for logarithms (property 7)

or $\frac{4}{5}\log_b x - \frac{1}{5}\log_b y - \frac{3}{5}\log_b z$

(ii) Natural Logarithmic function (Ln x)

$y = Ln\,x \Leftrightarrow X = \overset{x}{e}\ ,\ e = 2.7$

Domain: $(0, \infty)$

Range: R

Rewrite the logarithmic equations in exponential form.

a. $\log_2 32 = 5$ **b.** $\log_{10}\left(\frac{1}{1000}\right) = -3$ **c.** $\log_5 1 = 0$

	Logarithmic Form		Exponential Form
a.	$\log_2 32 = 5$	$\Leftrightarrow$	$2^5 = 32$
b.	$\log_{10}\left(\frac{1}{1000}\right) = -3$	$\Leftrightarrow$	$10^{-3} = \frac{1}{1000}$
c.	$\log_5 1 = 0$	$\Leftrightarrow$	$5^0 = 1$

Properties of the Natural Logarithmic Function

Let x and y be positive real numbers, and let p be a real number. Then the following properties are true.

1. $\ln 1 = 0$	**5.** $\ln(xy) = \ln x + \ln y$	Product property for logarithms
2. $\ln e = 1$	**6.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$	Quotient property for logarithms
3. $\ln e^p = p$	**7.** $\ln x^p = p \ln x$	Power property for logarithms
4. $e^{\ln x} = x$		

Simplify the expressions. Assume that all variable expressions within the logarithms represent positive real numbers.

a. $\ln e$ **b.** $\ln 1$ **c.** $\ln(e^{x+1})$ **d.** $e^{\ln(x+1)}$

Solution:

a. $\ln e = 1$	Property 2
b. $\ln 1 = 0$	Property 1
c. $\ln(e^{x+1}) = x + 1$	Property 3
d. $e^{\ln(x+1)} = x + 1$	Property 4

The Continuity :

Suppose f is a real function on a subset of the real numbers and let c be a point in the domain of f. Then, f is continuous at $x = c$, if

$$\lim_{x \to c} f(x) = f(c)$$

i.e. Left hand limit, right hand limit and the value of the function at $x = c$ exist and equal to each other.

In other words, a function is continuous at $x = c$, if the function is defined at $x = c$ and if the value of the function at $x = c$ equals the limit of the function at $x = c$.

Note If a function is not continuous at $x = c$, then it is said to be discontinuous at c and c is called the point of discontinuity of the function.

5.1.5 *Continuity of some of the common functions*

Function $f(x)$	Interval in which f is continuous
1. The constant function, i.e. $f(x) = c$	
2. The identity function, i.e. $f(x) = x$	**R**
3. The polynomial function, i.e. $f(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$	
4. $\lvert x - a \rvert$	$(-\infty, \infty)$
5. x^{-n}, n is a positive integer	$(-\infty, \infty) - \{0\}$
6. $p(x) / q(x)$, where $p(x)$ and $q(x)$ are polynomials in x	$\mathbf{R} - \{x : q(x) = 0\}$
7. $\sin x$, $\cos x$	**R**
8. $\tan x$, $\sec x$	$\mathbf{R} - \{ (2n+1) \frac{\pi}{2} : n \in \mathbf{Z}\}$
9. $\cot x$, $\operatorname{cosec} x$	$\mathbf{R} - \{ (n\pi : n \in \mathbf{Z}\}$

10. e^x **R**

11. $\log x$ $(0, \infty)$

The Derivative:

If $y = f(x)$ then the derivative is defined to be $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

If $y = f(x)$ then all of the following are equivalent notations for the derivative.

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}(f(x)) = Df(x)$$

If $y = f(x)$ all of the following are equivalent notations for derivative evaluated at $x = a$.

$$f'(a) = y'\big|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{dy}{dx}\bigg|_{x=a} = Df(a)$$

Basic Properties and Formulas

If $f(x)$ and $g(x)$ are differentiable functions (the derivative exists), c and n are any real numbers,

1. $(cf)' = cf'(x)$
2. $(f \pm g)' = f'(x) \pm g'(x)$
3. $(fg)' = f'g + fg'$ – **Product Rule**
4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$ – **Quotient Rule**
5. $\frac{d}{dx}(c) = 0$
6. $\frac{d}{dx}(x^n) = nx^{n-1}$ – **Power Rule**
7. $\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$

   This is the **Chain Rule**

Chain Rule Variants

The chain rule applied to some specific functions.

1. $\frac{d}{dx}\left(\left[f(x)\right]^n\right) = n\left[f(x)\right]^{n-1} f'(x)$
2. $\frac{d}{dx}\left(\mathbf{e}^{f(x)}\right) = f'(x)\mathbf{e}^{f(x)}$
3. $\frac{d}{dx}\left(\ln\left[f(x)\right]\right) = \frac{f'(x)}{f(x)}$
4. $\frac{d}{dx}\left(\sin\left[f(x)\right]\right) = f'(x)\cos\left[f(x)\right]$
5. $\frac{d}{dx}\left(\cos\left[f(x)\right]\right) = -f'(x)\sin\left[f(x)\right]$
6. $\frac{d}{dx}\left(\tan\left[f(x)\right]\right) = f'(x)\sec^2\left[f(x)\right]$
7. $\frac{d}{dx}\left(\sec\left[f(x)\right]\right) = f'(x)\sec\left[f(x)\right]\tan\left[f(x)\right]$
8. $\frac{d}{dx}\left(\tan^{-1}\left[f(x)\right]\right) = \frac{f'(x)}{1+\left[f(x)\right]^2}$

	$y = f(x)$	$\frac{dy}{dx}$
1	x^n	nx^{n-1}
2	e^x	e^x
3	e^{kx}	ke^{kx}
4	a^x	$a^x . \ln a$
5	$\ln x$	$\frac{1}{x}$
6	$\log_a x$	$\frac{1}{x . \ln a}$
7	$\sin x$	$\cos x$
8	$\cos x$	$-\sin x$
9	$\tan x$	$\sec^2 x$
10	$\cot x$	$-\mathrm{cosec}^2 x$
11	$\sec x$	$\sec x . \tan x$
12	$\mathrm{cosec}\, x$	$-\mathrm{cosec}\, x . \cot x$
13	$\sinh x$	$\cosh x$
14	$\cosh x$	$\sinh x$

1 $y = (4x - 5)^6$ $\qquad \dfrac{dy}{dx} = 6(4x-5)^5.4 = 24(4x-5)^5$

2 $y = e^{3-x}$ $\qquad \dfrac{dy}{dx} = e^{3-x}(-1) = -e^{3-x}$

3 $y = \sin 2x$ $\qquad \dfrac{dy}{dx} = \cos 2x.2 = 2\cos 2x$

4 $y = \cos(x^2)$ $\qquad \dfrac{dy}{dx} = -\sin(x^2).2x = -2x\sin(x^2)$

5 $y = \ln(3 - 4\cos x)$ $\qquad \dfrac{dy}{dx} = \dfrac{1}{3-4\cos x}.(4\sin x) = \dfrac{4\sin x}{3-4\cos x}$

1 $y = x^2 \tan x$ $\quad \therefore \dfrac{dy}{dx} = x^2 \sec^2 x + 2x \tan x$

$$= x(x \sec^2 x + 2 \tan x)$$

2 $y = e^{5x}(3x + 1)$ $\quad \therefore \dfrac{dy}{dx} = e^{5x}.3 + 5e^{5x}(3x + 1)$

$$= e^{5x}(3 + 15x + 5) = e^{5x}(8 + 15x)$$

3 $y = x \cos 2x$ $\quad \therefore \dfrac{dy}{dx} = x(-2 \sin 2x) + 1. \cos 2x$

$$= \cos 2x - 2x \sin 2x$$

4 $y = x^3 \sin 5x$ $\quad \therefore \dfrac{dy}{dx} = x^3.5 \cos 5x + 3x^2 \sin 5x$

$$= x^2(5x \cos 5x + 3 \sin 5x)$$

5 $y = x^2 \ln \sinh x$ $\quad \therefore \dfrac{dy}{dx} = x^2 \dfrac{1}{\sinh x} \cosh x + 2x \ln \sinh x$

$$= x(x \coth x + 2 \ln \sinh x)$$

Higher Order Derivatives

The Second Derivative is denoted as

$$f''(x) = f^{(2)}(x) = \frac{d^2 f}{dx^2}$$ and is defined as

$$f''(x) = \left(f'(x)\right)'$$, *i.e.* the derivative of the first derivative, $f'(x)$.

The n[th] Derivative is denoted as

$$f^{(n)}(x) = \frac{d^n f}{dx^n}$$ and is defined as

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$$, *i.e.* the derivative of the $(n\text{-}1)^{st}$ derivative, $f^{(n-1)}(x)$.

Implicit Functions :

If $y = x^2 - 4x + 2$, y is completely defined in terms of x, and y is called an *explicit function* of x.

When the relationship between x and y is more involved, it may not be possible (or desirable) to separate y completely on the left-hand side, e.g. $xy + \sin y = 2$. In such a case as this, y is called an *implicit function* of x, because a relationship of the form $y = f(x)$ is implied in the given equation.

It may still be necessary to determine the derivatives of y with respect to x and in fact this is not at all difficult. All we have to remember is that y is a function of x, even if it is difficult to see what it is. In fact, this is really an extension of our 'function of a function' routine.

$x^2 + y^2 = 25$, as it stands is an example of an . . . . function.

Once again, all we have to remember is that y is a function of x. So, if $x^2 + y^2 = 25$, let us find $\frac{dy}{dx}$

If we differentiate as it stands with respect to x, we get

$$2x + 2y\frac{dy}{dx} = 0$$

Note that we differentiate y^2 as a function squared, giving 'twice times the function, times the derivative of the function'. The rest is easy.

$$2x + 2y\frac{dy}{dx} = 0$$

$$\therefore\ y\frac{dy}{dx} = -x \qquad \therefore\ \frac{dy}{dx} = -\frac{x}{y}$$

Let us look at some examples.

If $x^2 + y^2 - 2x - 6y + 5 = 0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 3$, $y = 2$.

Differentiate as it stands with respect to x.

$$2x + 2y\frac{dy}{dx} - 2 - 6\frac{dy}{dx} = 0$$

$$\therefore\ (2y - 6)\frac{dy}{dx} = 2 - 2x \qquad \therefore\ \frac{dy}{dx} = \frac{2 - 2x}{2y - 6} = \frac{1 - x}{y - 3}$$

$$\therefore\ \text{at } (3, 2) \qquad \frac{dy}{dx} = \frac{1 - 3}{2 - 3} = \frac{-2}{-1} = 2$$

$$\text{Then} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left\{\frac{1 - x}{y - 3}\right\} = \frac{(y - 3)(-1) - (1 - x)\frac{dy}{dx}}{(y - 3)^2}$$

$$= \frac{(3 - y) - (1 - x)\frac{dy}{dx}}{(y - 3)^2}$$

$$\text{at } (3, 2) \qquad \frac{d^2y}{dx^2} = \frac{(3 - 2) - (1 - 3)2}{(2 - 3)^2} = \frac{1 - (-4)}{1} = 5$$

$$\therefore\ \text{At } (3, 2) \quad \frac{dy}{dx} = 2,\ \frac{d^2y}{dx^2} = 5$$

The Integration :

Integration is considered as the anti derivative operation.

The formula $\frac{d}{dx}$ F(x)=f(x)

is the same as $\int f(x)dx = F(x) + c$

ex: $\frac{d}{dx}\sin x = \cos x$ $\quad\int \cos x = \sin x + c$

Rules:

1. $\int dx = x + c$ 2. $\int kdx = kx + c$,k is constant

3.$\int x^n dx = \frac{x^{n+1}}{n+1} + c$ n$\neq -1$

4.$\int \big(f(x) + g(x)\big)dx = \int f(x)dx + \int g(x)dx$

5. $\int kf(x) = k\int f(x)\,dx$, k is constant

6. $\int f(x)^n f'(x)dx = \frac{f(x)^{n+1}}{n+1} + c$

Ex:

1. $\int 2dx = 2x + c$

2. $\int x^3 dx = \frac{x^4}{4} + c$

3. $\int -2x^2 dx = \frac{-2x^3}{3} + c$

4. $\int (x^2 + 2)^2 . 2xdx = \frac{(x^2+2)^3}{3} + c$

5. $\int (x^3 - 1)^3 . x^2 dx = \frac{1}{3} \int (x^3 - 1)^3 . 3x^2 dx = \frac{(x^3-1)^4}{12} + c$

The Trigonometric functions

1. $\int \sin x \, dx = -\cos x + c$

2. $\int \cos x \, dx = \sin x + c$

3. $\int sec^2 xdx = \tan x + c$

4. $\int csc^2 xdx = -cotx + c$

5. $\int sec\ xtanxdx = \sec x + c$

6. $\int csc\ xcotxdx = -\csc x + c$

EX:

1.$\int xcos(x^2+1)dx = \frac{1}{2}\int 2xcos(x^2+1)dx = \frac{1}{2}\sin(x^2+1)+c$

2. $\int sec^2(4x)\,dx = \frac{\tan(4x)}{4}+c$

3.$\int \cot x\, sin^3 x\, dx = \int \frac{\cos x}{\sin x} sin^3 x dx$

$= \int \cos x\, sin^2 x\, dx = \frac{sin^3 x}{3}$ +c

أما في حالة عدم توفر المشتقة فنتبع الأتي :

1.if power is even :

$$cos^2 x = \frac{(1+\cos 2x)}{2} \qquad sin^2 x = \frac{(1-\cos 2x)}{2}$$

2.if power is odd:

$$sin^2 x = 1 - cos^2 x \qquad cos^2 x = 1 - sin^2 x$$

$$sec^2 x = 1 + tan^2 x$$

EX:

$$\int cos^3 x \;\; dx = \int \cos x\, cos^2 x\, dx = \int \cos x(1$$

Integration of multiply sin and cos with diff. angles

$$1.\int sin\ ax cos\ bx dx = \frac{1}{2}\int(\sin(a-b)\,x + \sin(a+b)\,x)dx$$

$$2.\int sin\ ax sin\ bx dx = \frac{1}{2}\int(\cos(a-b)\,x - \cos(a+b)\,x)dx$$

$$3.\int cos\ ax cos\ bx dx = \frac{1}{2}\int(\cos(a-b)\,x + \cos(a+b)\,x)dx$$

EX

$$1.\int sin\ 4x cos 2x dx = \frac{1}{2}\int(\sin(4-2)\,x + \sin(4+2)\,x)dx =$$

$$\frac{1}{2}\int(\sin 2\,x + \sin 6\,x)dx = \frac{1}{2}\left(-\frac{1}{2}cos2x - \frac{1}{6}cos6x\right) + c$$

The Logarithmic functions

1. Natural logarithm of x

$$\frac{d}{dx}lnx = \frac{1}{x} \qquad \therefore \int \frac{1}{x}dx = ln|x| + c$$

2.e^x function

$$\frac{d}{dx}e^x = e^x \qquad\qquad \therefore \int e^x\,dx = e^x + c$$

3. a^x function

$\frac{d}{dx}a^x = a^x lna \qquad\qquad \therefore \int a^x\,dx = \frac{a}{lna}^x + c$

Ex:

1.

$\int \frac{2x}{x^2}dx = ln|x^2| + c$

2.$\int e^{3x}dx = \frac{1}{3}\int 3e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

3.$\int 2^{4x}\,dx = \frac{1}{4}\frac{2}{ln2}^x + c$

$$\therefore \int \tan x\,dx = -ln|\cos x| + c$$

$$\int \cot x\,dx = ln|\sin x| + c$$

The Hyperbolic Functions

1. $\int \sinh x\, dx = \cos hx + c$ 2. $\int \cosh x\, dx = \sin hx + c$

3. $\int sech^2 x dx = \tanh x + c$ 4. $\int csch^2 x dx = -cothx + c$

5. $\int sec\ hxtanhxdx = -\sec hx + c$

6. $\int csch\ xcothxdx = -\csc hx + c$

7. $\int \tanh x\, dx = ln|\cosh x| + c$ 8. $\int \coth x\, dx = ln|\sinh x| + c$

Ex:

1. $\int \frac{dx}{sinhx+coshx} = \int \frac{dx}{e^x} = \int e^{-x} dx = -e^{-x} + c$

2. $\int e^{2x} sinh3xdx = \int e^{2x} (\frac{e^{3x}-e^{-3x}}{2}) dx = \frac{1}{2}(\int (e^{5x} - e^{-x}) dx = \frac{1}{10} e^{5x} + \frac{1}{2} e^{-x} + c$

Here is a list of basic derivatives and the basic integrals that go with them:

1	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\therefore \int x^n \, dx = \frac{x^{n+1}}{n+1} + C$	$\left\{ \begin{array}{c} \text{provided} \\ n \neq -1 \end{array} \right\}$
2	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\therefore \int \frac{1}{x} \, dx = \ln x + C$	
3	$\frac{d}{dx}(e^x) = e^x$	$\therefore \int e^x \, dx = e^x + C$	
4	$\frac{d}{dx}(e^{kx}) = ke^{kx}$	$\therefore \int e^{kx} \, dx = \frac{e^{kx}}{k} + C$	
5	$\frac{d}{dx}(a^x) = a^x \ln a$	$\therefore \int a^x \, dx = \frac{a^x}{\ln a} + C$	
6	$\frac{d}{dx}(\cos x) = -\sin x$	$\therefore \int \sin x \, dx = -\cos x + C$	
7	$\frac{d}{dx}(\sin x) = \cos x$	$\therefore \int \cos x \, dx = \sin x + C$	
8	$\frac{d}{dx}(\tan x) = \sec^2 x$	$\therefore \int \sec^2 x \, dx = \tan x + C$	
9	$\frac{d}{dx}(\cosh x) = \sinh x$	$\therefore \int \sinh x \, dx = \cosh x + C$	
10	$\frac{d}{dx}(\sinh x) = \cosh x$	$\therefore \int \cosh x \, dx = \sinh x + C$	

1 $\int (2x-7)^3 \, dx = \frac{(2x-7)^4}{2.4} + C = \frac{(2x-7)^4}{8} + C$

2 $\int \cos(7x+2) \, dx = \frac{\sin(7x+2)}{7} + C$

3 $\int e^{5x+4} \, dx = \frac{e^{5x+4}}{5} + C$

4 $\int \sinh 7x \, dx = \frac{\cosh 7x}{7} + C$

5 $\int \frac{1}{4x+3} \, dx = \frac{\ln(4x+3)}{4} + C$

7 $\int \sec^2(3x+1) \, dx = \frac{\tan(3x+1)}{3} + C$

8 $\int \sin(2x-5) \, dx = -\frac{\cos(2x-5)}{2} + C$

9 $\int \cosh(1+4x) \, dx = \frac{\sinh(1+4x)}{4} + C$

10 $\int 3^{5x} \, dx = \frac{3^{5x}}{5 \ln 3} + C$

Method of Integral :

1-. Integration by Substitution:

EX:

1.$\int x(x^2+1)^{20}dx$

Let u=$(x^2+1) \rightarrow x^2 = u-1 \quad \rightarrow$ x=$\sqrt{u-1}$

du=2xdx dx=$\frac{du}{2x}$ dx=$\frac{du}{2\sqrt{u-1}}$

$$\int \sqrt{u-1}\,(u)^{20}\frac{du}{2\sqrt{u-1}} = \frac{1}{2}\int (u)^{20}du = \frac{(u)^{21}}{42} + c$$

2. $\int x\sqrt{x-3}\,dx =$

u=x-3 x=u+3 du=dx

$$\int (u+3)\sqrt{u}\,du$$

$$= \int \left((u)^{3/2} + 3(u)^{1/2}\right)du \quad = \frac{(u)^{5/2}}{\frac{5}{2}}$$

$$+ 3\frac{(u)^{3/2}}{\frac{3}{2}} + c = \frac{2}{5}(u)^{5/2}+2(u)^{3/2}+c$$

2-.Integration by parts(udv)

$$\int udv = uv - \int vdu$$

1.$\int xe^{3x}dx =$

u=x du=dx

dv=$e^{3x}dx$ $v = \frac{1}{3}e^{3x}$

$\therefore \int udv = \frac{1}{3}xe^{3x} - \int \frac{1}{3}e^{3x}dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + c =$

$\frac{1}{3}e^{3x}(x - \frac{1}{3}) + c$

EX:

(a) $$\int x \ln x \, dx = \ln x\left(\frac{x^2}{2}\right) - \frac{1}{2}\int x^2 . \frac{1}{x}\, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2}\int x \, dx$$

$$= \frac{x^2 \ln x}{2} - \frac{1}{2} . \frac{x^2}{2} + C$$

$$= \frac{x^2}{2}\left\{\ln x - \frac{1}{2}\right\} + C$$

(b) $$\int x^3 e^{2x}\, dx = x^3\left(\frac{e^{2x}}{2}\right) - \frac{3}{2}\int e^{2x} x^2 \, dx$$

$$= \frac{x^3 e^{2x}}{2} - \frac{3}{2}\left\{x^2\left(\frac{e^{2x}}{2}\right) - \frac{2}{2}\int e^{2x} x \, dx\right\}$$

$$= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3}{2}\left\{x\left(\frac{e^{2x}}{2}\right) - \frac{1}{2}\int e^{2x}\, dx\right\}$$

$$= \frac{x^3 e^{2x}}{2} - \frac{3x^2 e^{2x}}{4} + \frac{3x e^{2x}}{4} - \frac{3}{4}\,\frac{e^{2x}}{2} + C$$

$$= \frac{e^{2x}}{2}\left\{x^3 - \frac{3x^2}{2} + \frac{3x}{2} - \frac{3}{4}\right\} + C$$

The Definite Integral :

Theorem: (Fundamental Theorem I):

$$\int_a^b f(x)dx = F(x)]_a^b = F(b) - F(a) \qquad , \frac{d}{dx}F(x) = f(x)$$

EX:

1. Find $\int_2^3 x^2 dx = \frac{x^3}{3}]_2^3 = \frac{3^3}{3} - \frac{2^3}{3} = \frac{27}{3} - \frac{8}{3} = \frac{19}{3}$

2. $\int_0^\pi \sin^2 x\, dx = \int_0^\pi \frac{1}{2}(1 - \cos 2x)dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)]_0^\pi = \frac{1}{2}\left(\pi - \frac{\sin 2\pi}{2}\right) - \frac{1}{2}\left(0\right.$

a.

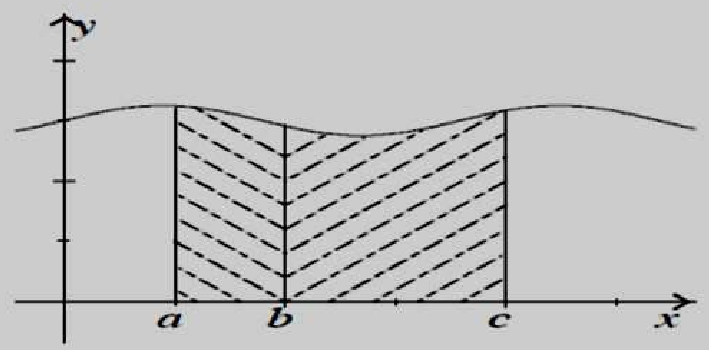
$$\int_a^a f(x)dx = 0.$$

If the upper and lower limits of the integral are the same, the integral is zero. This becomes obvious if we have a positive function and can interpret the integral in terms of 'the area under a curve'.

b. If $a \le b \le c$,

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

This says that the integral of a function over the union of two intervals is equal to the sum of the integrals over each of the intervals. The diagram opposite helps to make this clear if $f(x)$ is a positive function.



c.

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx \quad \text{for any constant } c.$$

This tells us that we can move a constant past the integral sign, but *beware*: we can only do this with constants, never with variables!

d.

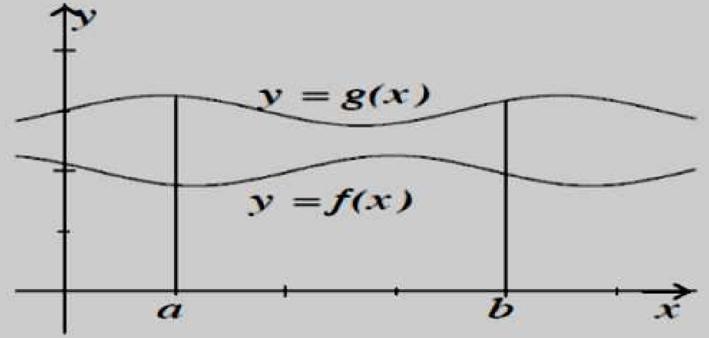
$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx.$$

That is, the integral of a sum is equal to the sum of the integrals.

e. If $f(x) \le g(x)$ in $[a, b]$ then

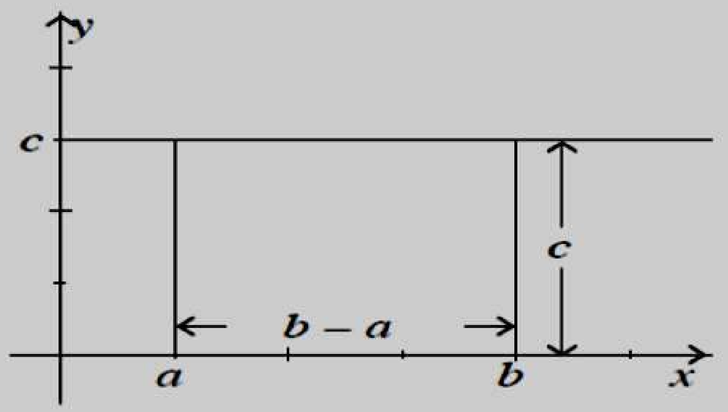
$$\int_a^b f(x)dx \le \int_a^b g(x)dx.$$

That is, integration preserves inequalities between functions. The diagram opposite explains this result if $f(x)$ and $g(x)$ are positive functions.



f.
$$\int_a^b c\,dx = c(b-a).$$

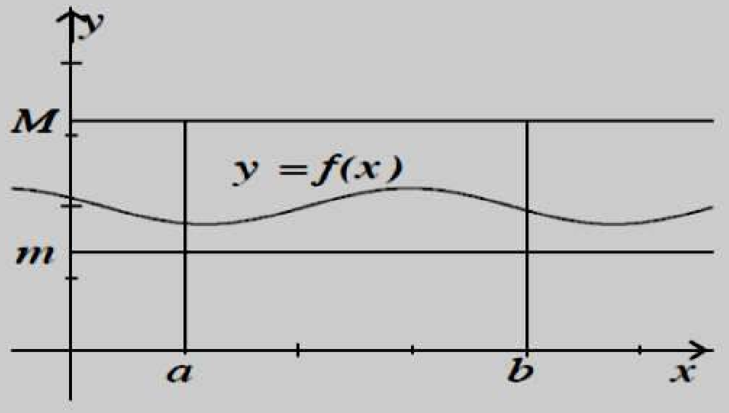
This tells us that the integral of a constant is equal to the product of the constant and the range of integration. It becomes obvious when we look at the diagram with $c > 0$, since the area represented by the integral is just a rectangle of height c and width $b - a$.



g. We can combine (e) and (f) to give the result that, if M is any upper bound and m any lower bound for $f(x)$ in the interval $[a, b]$, so that $m \le f(x) \le M$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a).$$

This, too, becomes clear when $f(x)$ is a positive function and we can interpret the integral as the area under the curve.



h. Finally we extend the definition of the definite integral slightly, to remove the restriction that the lower limit of the integral must be a smaller number than the upper limit. We do this by specifying that

$$\int_b^a f(x)dx = -\int_a^b f(x)dx.$$

For example,

$$\int_2^1 f(x)dx = -\int_1^2 f(x)dx.$$

The applications of definite integral

1-Area under the curve

2.Area between two curves

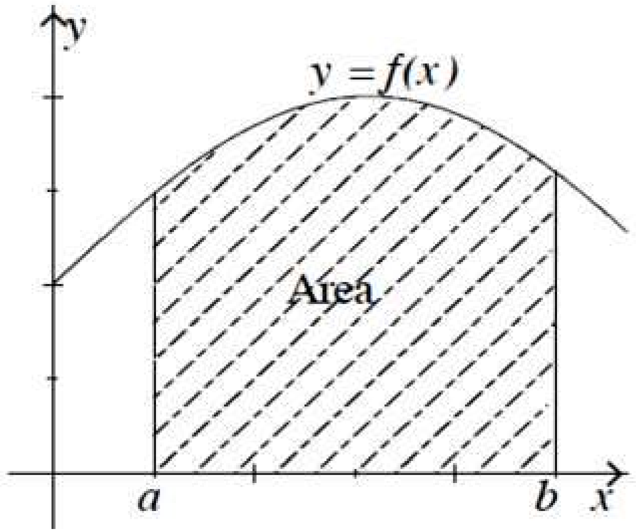
3.Volumes

4 The length of the curves

5. Surface area

1. **Area under the curve** The integral $\int_a^b f(x)dx$ is defined the area under the curve y=f(x) from x=a to x=b

A=$\int_a^b y\,dx$

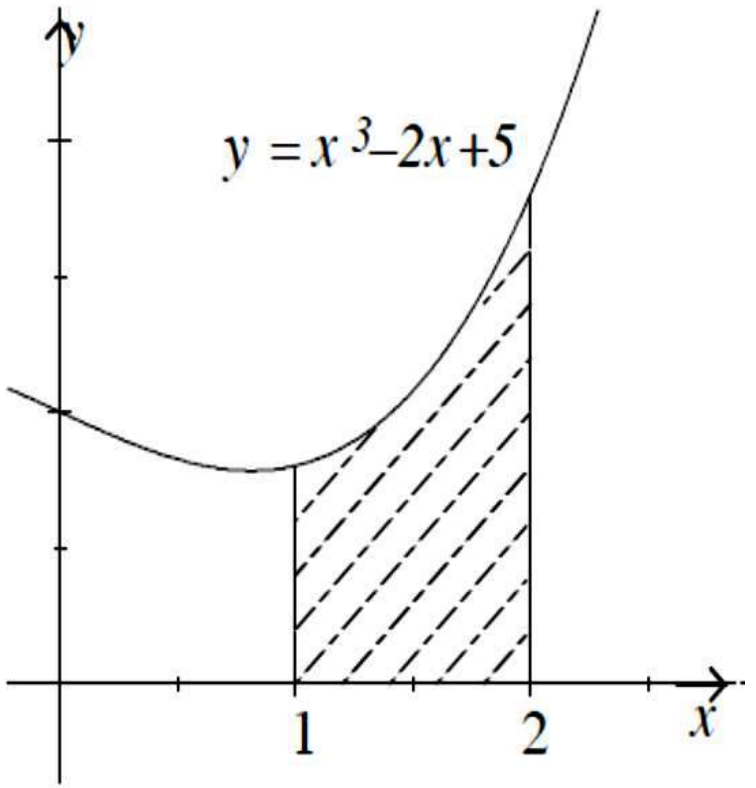


Find the area enclosed between the x-axis, the curve $y = x^3 - 2x + 5$ and the ordinates $x = 1$ and $x = 2$.

In a question like this it is always a good idea to draw a rough sketch of the graph of the function and the area you are asked to find. (See below)

If the required area is A square units, then

$$\begin{aligned} A &= \int_1^2 \left(x^3 - 2x + 5\right) dx \\ &= \left[\frac{x^4}{4} - x^2 + 5x\right]_1^2 \\ &= (4 - 4 + 10) - \left(\frac{1}{4} - 1 + 5\right) \\ &= 5\frac{3}{4}. \end{aligned}$$

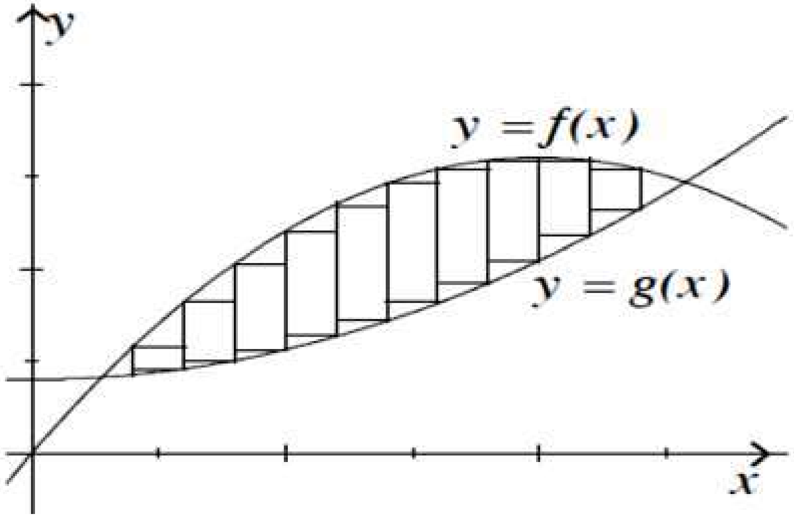


2. **Area between two curves** :

Case 1(No intersecting of two curves in [a,b])

If $y_1 = f(x), y_2 = g(x)$ then the area is defined as

$A_a^b = \int_a^b |y_1 - y_2| dx$



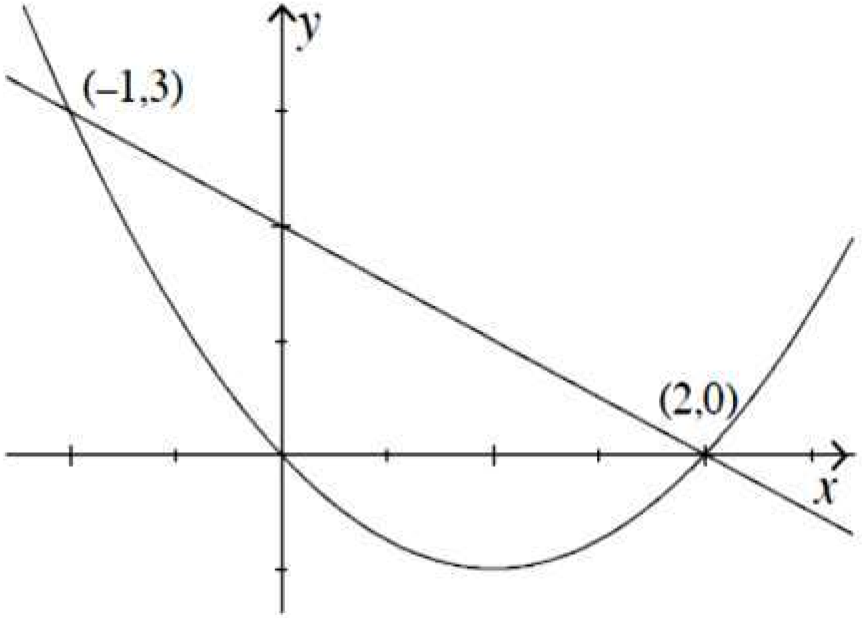
Case 2(An intersecting point exists in [a,b])

The area between two curves y_1, y_2 from x=a to x=b is defined as

$$A_a^b = \int_a^c (y_2 - y_1) dx + \int_c^b (y_1 - y_2)\, dx$$

Find the area enclosed between the parabola $y = x(x-2)$ and the line $y = -x + 2$.

The curves $y = x^2 - 2x$ and $y = -x + 2$ intersect where $x^2 - 2x = -x + 2$. i.e. at $x = -1$ or $x = 2$.
The upper curve is $y = -x + 2$.



$$
\begin{aligned}
\text{Area} &= \int_{-1}^{2} ((-x+2) - (x^2 - 2x))dx \\
&= \int_{-1}^{2} (2 + x - x^2)dx \\
&= \left[2x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_{-1}^{2} \\
&= (4 + 2 - \frac{8}{3}) - (-2 + \frac{1}{2} + \frac{1}{3}) \\
&= 4\frac{1}{2}.
\end{aligned}
$$

i Sketch the graphs of the function $y = 6 - x - x^2$ and $y = x^3 - 7x + 6$.

ii Find the points of intersection of the curves.

iii Find the total area enclosed between them.

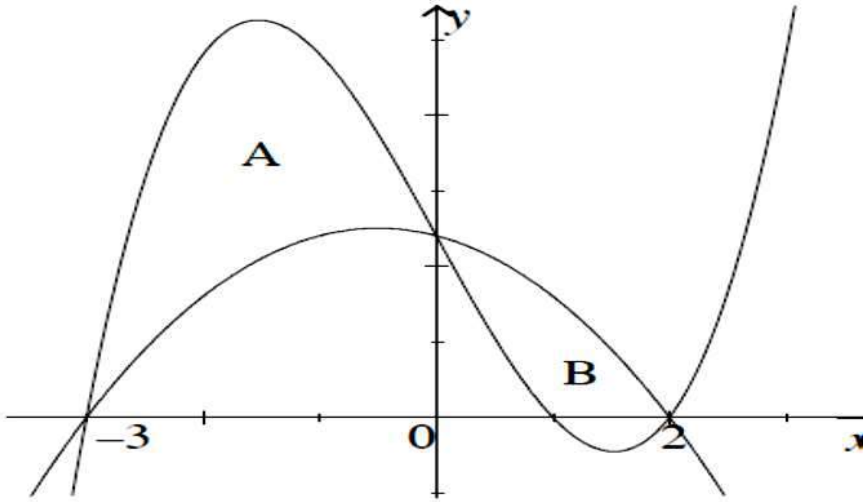
(i) and (ii) The curves are easier to sketch if we first find the points of intersection: they meet where $x^3 - 7x + 6 = 6 - x - x^2$.
That is,

$$x^3 + x^2 - 6x = 0$$

or

$$x(x-2)(x+3) = 0.$$

So the points of intersection are $(0,6)$; $(2,0)$; and $(-3,0)$.
The first curve is an 'upside-down' parabola, and the second a cubic.
Total area = area A + area B.



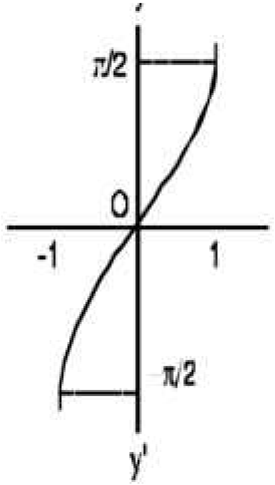
$$\begin{aligned}
\text{Area A} &= \int_{-3}^{0}((x^3 - 7x + 6) - (6 - x - x^2))dx \\
&= \int_{-3}^{0}(x^3 + x^2 - 6x)dx \\
&= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2\right]_{-3}^{0} \\
&= 15\frac{3}{4}.
\end{aligned}$$

$$\begin{aligned}
\text{Area B} &= \int_{0}^{2}((6 - x - x^2) - (x^3 - 7x + 6))dx \\
&= \int_{0}^{2}(6x - x^2 - x^3)dx \\
&= 5\frac{1}{3}.
\end{aligned}$$

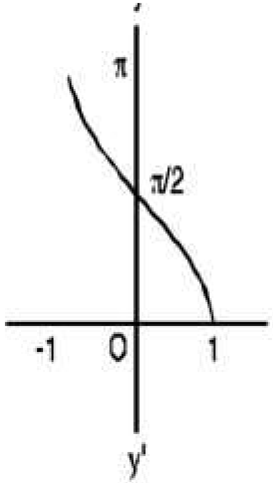
$\therefore$ the total area $= 15\frac{3}{4} + 5\frac{1}{3} = 21\frac{1}{12}$ square units.

Inverse of Trigonometric Functions:

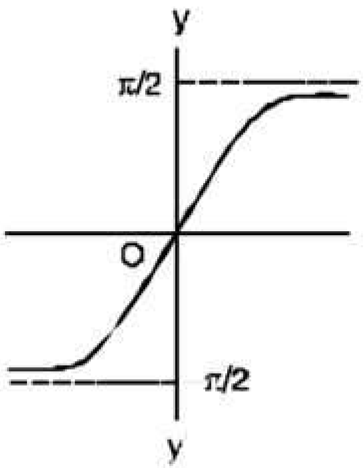
Function	Domain	Range (Principal value)
$y = \sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$y = \tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \cot^{-1} x$	R	$[0, \pi]$
$y = \sec^{-1} x$	$x \geq 1$ or $x \leq -1$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
$y = \operatorname{cosec}^{-1} x$	$x \geq 1$ or $x \leq -1$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$



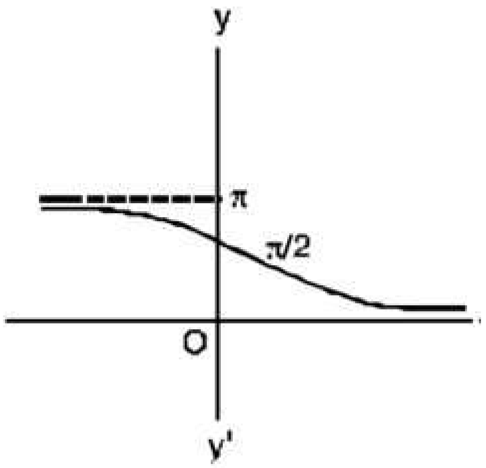
$y = \sin^{-1} x$



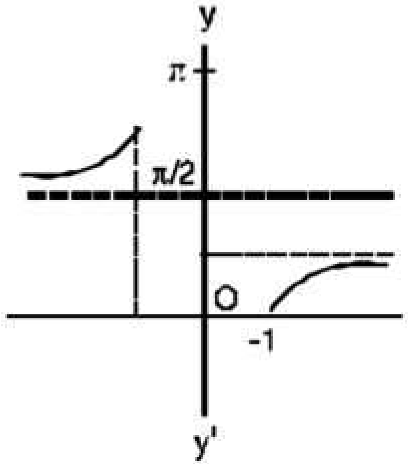
$y = \cos^{-1} x$



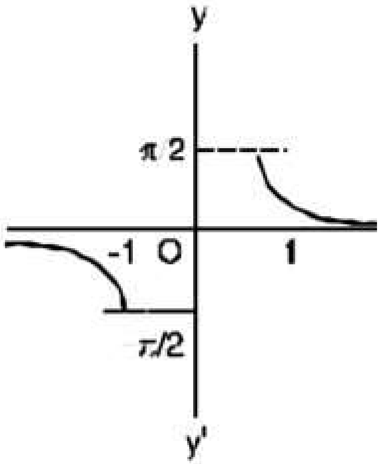
$y = \tan^{-1} x$



$y = \cot^{-1} x$



$y = \sec^{-1} x$



$y = \text{cosec}^{-1} x$

Derivatives of Inverse Trig Functions

- $\frac{d}{dx}(\sin^{-1}x)=\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}$
- $\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}$
- $\frac{d}{dx}(\csc^{-1}x)=-\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}$
- $\frac{d}{dx}(\tan^{-1}x)=-\frac{1}{1+x^2}$

Useful Integration Formulas

- $\int \frac{1}{\sqrt{1-x^2}}\, dx$ =$\sin^{-1}x$ + C (1)

- $\int \frac{1}{x^2+1}\, dx$ = $\tan^{-1}x$ + C (2)

- $\int \frac{1}{x2+a2}\, dx = \frac{1}{a}\ tan^{-1}(\frac{x}{a})$ +C (3)

EX:

1. $\int \frac{x^2}{1+x^6} dx = \int \frac{x^2}{1+(x^3)^2} dx = \frac{1}{3}\int \frac{3x^2}{1+(x^3)^2} dx = \frac{1}{3} tan^{-1} x^3 + c$

2. $\int \frac{x}{\sqrt{1-x^4}} dx = \frac{1}{2}\int \frac{2x}{\sqrt{1-(x^2)^2}} dx = \frac{1}{2} sin^{-1} x^2 + c$

3- $\int x sec^{-1} x \, dx =$

u = $\sec^{-1} x$ $\quad du = \frac{dx}{x\sqrt{x^2-1}}$

dv = xdx $\quad$ v = $\frac{x^2}{2}$

$\therefore \int x sec^{-1} x \, dx = \sec^{-1} x \frac{x^2}{2} - \int \frac{x^2}{2} \frac{dx}{x\sqrt{x^2-1}} =$

$\sec^{-1} x \frac{x^2}{2} - \frac{1}{2}\int \frac{xdx}{x\sqrt{x^2-1}}$

$= \sec^{-1} x \frac{x^2}{2} - \frac{1}{2} * \frac{1}{2} * \frac{(x^2-1)^{1/2}}{1/2}$ +c= $\sec^{-1} x \frac{x^2}{2} - \frac{1}{2} * \sqrt{x^2 - 1} + c$

4-.$\int \tan^{-1} x\, dx =$

u= $\tan^{-1} x$ du= $\frac{dx}{1+x^2}$

dv= dx v= x

$$\int \tan^{-1} x\, dx = \tan^{-1} x * x - \int x \frac{dx}{1+x^2} = \tan^{-1} x * x - \frac{1}{2}\, ln|1+x^2| + c$$